SOLUTIONS NOVEMBER 2020

ACTIVITIES FOR 3ESO AND 4ESO. 14-16 YEARS. AUTHORS: COLLECTIVE "CONCURSO DE PRIMAVERA"

https://www.concursoprimavera.es/#concurso

November 2: If x + y = 18 and $x^2 + y^2 = 212$ calculate the value of:

$$|x^2 - y^2|$$

Solution: We have to solve the system:

$$\left.\begin{array}{r} x + y = 18 \\ x^2 + y^2 = 212 \end{array}\right\}$$

We have:

$$(x+y)^2 = \left\{\begin{array}{l} = 18^2 \\ = x^2 + y^2 + 2xy = 212 + 2xy \end{array}\right\} \Rightarrow \quad xy = 56$$

Therefore, x and y are solutions of the equation: $z^2 - 18z + 56 = 0$ ($z^2 - Sz + P = 0$)

By last:

$$z^2 - 18z + 56 = 0 \;\Rightarrow\; z = \frac{18 \pm \sqrt{18^2 - 4 \cdot 56}}{2} = \left\{\begin{array}{l} = 14 \\ = 4 \end{array}\right\} \;\Rightarrow\; |x^2 - y^2| = |14^2 - 4^2| = 180$$

November 3: 2020 is a multiple of 20, how many numbers of the form 2000 + b with b natural and less than 1000 are divisible by b?

Solution: As b|b we have; b|(2000 + b) ⇔ b|2000. Therefore, we look for the divisors of 2000 less than 1000. Since $2000 = 2^4 \cdot 5^3$, we will have that there are ((4 + 1) · (3 + 1) =) 20 divisors of 2000. From them we have to eliminate the greater or equal a 1000: 2000 and 1000. Then there are a total of (20 - 2 =) 18 possible values of b: 1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500.

November 4: All the products of 8 different factors are formed that can be generated with the digits from 1 to 9. In what digit does the sum of all the products end?

Solution: We consider the products of eight different factors from the numbers: 1, 2, 3, 4, 5, 6, 7, 8 and 9. These products can meet:

They do not contain the digit 2: So they contain the digits 4 and 5. Then they end in 0

They do not contain the digit 5: That addition ends in 6, because by grouping the products two by two:

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \xrightarrow{\text{ends in}} 2 \cdot 2 \cdot 2 \cdot 2 \xrightarrow{\text{ends in}} 4 \cdot 4 \xrightarrow{\text{ends in}} 6$$

Contains 2 and 5: They end in 0.

Then the sum considered ends in 6.

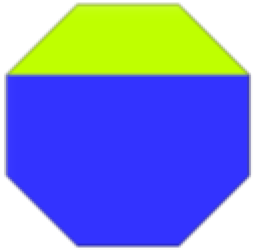
November 5: Find p, q, r∈ℕ, knowing that:

$$\mathbf{p} + \cfrac{\mathbf{1}}{\mathbf{q} + \cfrac{\mathbf{1}}{\mathbf{r}}} = \frac{\mathbf{25}}{\mathbf{19}}$$

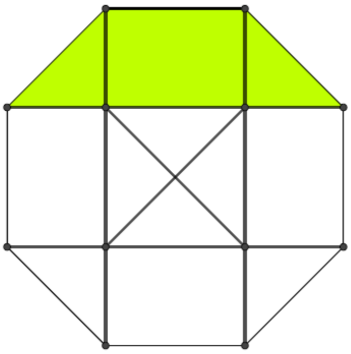
Solution: We have:

$$p+\cfrac{1}{q+\cfrac{1}{r}}=\frac{25}{19}=\frac{19+6}{19}=1+\frac{6}{19}=1+\cfrac{1}{\cfrac{19}{6}}=1+\cfrac{1}{\cfrac{6\cdot 3+1}{6}}=1+\cfrac{1}{3+\cfrac{1}{6}}\Rightarrow\begin{cases}p=1\\q=3\\r=6\end{cases}$$

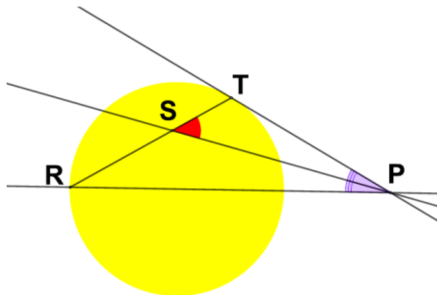
November 6-7: Calculate the area of the regular octagon of the figure knowing that the area of the green area is 3 cm^2



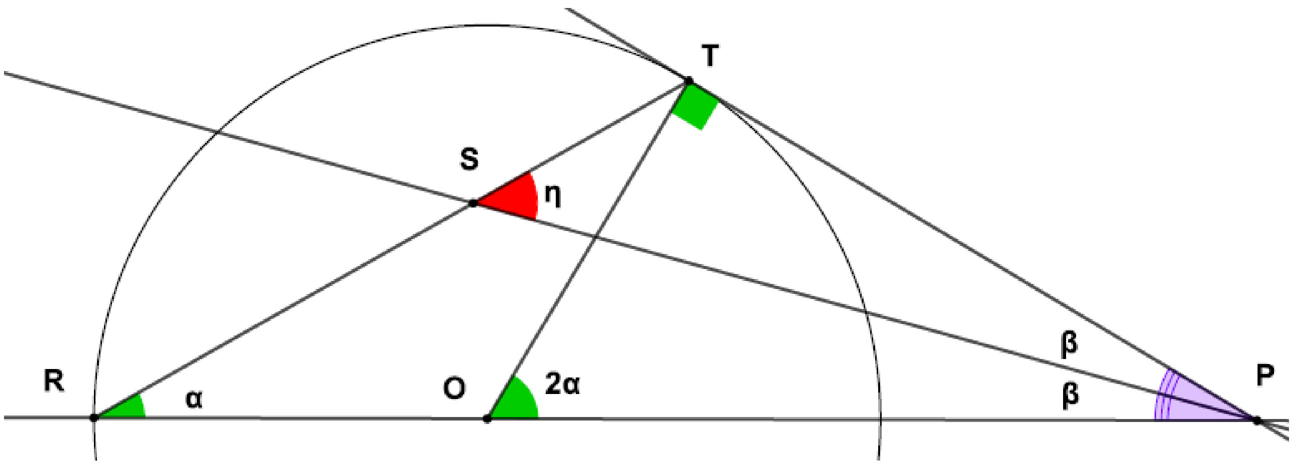
Solution: We draw the rectilinear segments of the attached figure. Then the initial octagon is divided into four equal rectangles, a central square and in the corners four isosceles right triangles that coincide with the four triangles into which the central square has been divided. In total eight equal triangles and four equal rectangles. But one of these rectangles together with two of these triangles have area 3. Then the area of the octagon is: (3 · 4 =) 12 cm^2



November 9-10: From a point P, outside a circle, draw two lines, PR passing through the centre of the circle and PT tangent to the circle at T. Let PS be the bisector of ∠RPT. Find ∠TSP



Solution:



In the first place, if O is the centre of the circumference (by virtue of the relationship between the central angle and the inscribed angle) we will have the angles marked in the figure. Also, in ΔOTP $2\alpha + 2\beta = 90°$, therefore $\alpha + \beta = 45°$.

Also, ΔRSO is isosceles, since RO = TO $\Rightarrow$ $\angle RTO = \alpha$.

Finally, in ΔSTP:

$$\eta + 90° + \alpha + \beta = 180° \quad (\alpha + \beta = 45°) \Rightarrow \eta = 45°$$

November 11: In the equation:

$$N\cdot U\cdot(M+E+R+0) = 33$$

each letter represents a different digit. In how many ways can we choose the value of the letters?

Solution: Since $33 = 1\cdot 3\cdot 11$ and neither N nor U can be equal to 11, we will have that, necessarily, the parentheses must be worth 11. Therefore, N and U must be worth, one 3 and the other 1 and must be fulfilled

$$M + E + R + O = 11$$

8	2	1	0	Repeats 1
7	3	1	0	Repeats 1 and 3
6	3	2	0	Repeats 3
5	3	2	1	Repeats 1 and 3
5	4	2	0	VALID

Hence M, E, R, O $\in\{5, 4, 2, 0\}$. Then there are 4! ways to assign values for M, E, R and O. Since there are also two ways to assign values for N and U, we will have that there are $(4! \cdot 2 =)$ 48 ways to choose the value of each letter

November 12-19: At 12:00 in the morning a train leaves from city A to B, and at 12:40 another one leaves from B to A. Both circulate at the same constant speed throughout the journey and take three and a half hours to do. the route, what time do they cross?

Solution: The journey lasts three and a half hours, that is $(3{,}5\cdot 60 =)$ 210 minutes. For 40 minutes only one train is traveling. Then at $(210 - 40 =)$ 170 the two trains are traveling. Since they travel at the same speed, each of them travels $(170/2 =)$ 85 minutes to the intersection. Then from 12:00 they have to pass $(40 + 85 =)$ 125 minutes, that is, 2 hours and 5 minutes until the crossing. That is, they will cross at 14:05.

November 13: If a, b and c are natural such that

$$\left(\frac{a}{c}+\frac{a}{b}+1\right):\left(\frac{b}{a}+\frac{b}{c}+1\right) = 11$$

How many triples do you verify:

$$a + 2b + c \leq 40?$$

Solution: We have:

$$\left(\frac{a}{c}+\frac{a}{b}+1\right):\left(\frac{b}{a}+\frac{b}{c}+1\right)=11=\frac{\frac{ba+ac+cb}{cb}}{\frac{bc+ba+ac}{ac}}=\frac{ac}{cb}\cdot\frac{b(a\ +c)+ac}{b(c+a)+ac}=\frac{a}{b}\Rightarrow a=11b$$

Therefore:

$$a+2b+c\leq 40\ \ \Rightarrow\ \ 13b+c\leq 40$$

Since b,c $\in\mathbb{N}$, we will have:

b = 1 $\Rightarrow$ c ≤ 27 $\Rightarrow$ $(a, b, c)\in\{(11, 1, c)\mid c\in\{1, 2, 3, \cdots, 27\}\}$ $\Rightarrow$ 27 triples

b = 2 $\Rightarrow$ c ≤ 14 $\Rightarrow$ $(a, b, c)\in\{(22, 2, c)\mid c\in\{1, 2, 3, \cdots, 14\}\}$ $\Rightarrow$ 14 triples

b = 3 $\Rightarrow$ c ≤ 1 $\Rightarrow$ $(a, b, c)=(33, 3, 1)$ $\Rightarrow$ 1 triple

n total (27 + 14 + 1 =) 42 triples

November 14: All of Alaska's oil reserves would last 35 years if only consumed by the US, if they were also consumed by China, they would last only 10 years. How many years would it last if only China consumed them?

Solution: Let r be the total reserves of Alaska, u the annual consumption of the United States and c the annual consumption of China. From the statement of the problem we have:

$$r=35u$$
$$r=10(u+c)$$

Solving for u from the first equation:

$$u=\frac{r}{35}$$

And substituting in the second:

$$r=10\left(\frac{r}{35}+c\right)=\frac{10r}{35}+10c\ \ \Rightarrow\ \ r-\frac{10r}{35}=10c\ \ \Rightarrow\ \ \frac{25r}{35}=10c\ \ \Rightarrow\ \ \frac{r}{c}=\frac{35\cdot 10}{25}=14$$

In other words, Alaska's oil reserves would last 14 years if only consumed by China.

November 16: If **|u – 10|= v** and **u < 10** What value does u - v take?

Solution 1: Being u < 10, we have u – 10 > 0. Therefore:

v = |u – 10|= 10 – u $\Rightarrow$ u – v = u – 10 + u = 2u – 10

Solution 2: Being u < 10, we have u – 10 > 0. Therefore:

v = |u – 10|= 10 – u $\Rightarrow$ u = 10 – v $\Rightarrow$ u – v = 10 – v – v = 10 – 2v

November 17: If B > A > 1, compare the fractions

$$\frac{A-1}{B-1},\qquad\frac{A+1}{B+1},\qquad\frac{A^2-1}{B^2-1},\qquad\frac{A^3-1}{B^3-1}$$

Solution: We have, if B > A > 1

$$(A+1)\cdot(B-1)=AB+B-A-1\{B-A>0>A-B\}>AB-B+A-1=(A-1)\cdot(B+1)$$

From where (since B + 1 > 0 and B – 1 > 0)

$$\frac{A+1}{B+1}>\frac{A-1}{B-1}$$

On the other hand:

$$B+1>A+1 \xRightarrow[B-1>A-1>0]{} \frac{(B+1)\cdot(B-1)}{(B-1)} > \frac{(A+1)\cdot(A-1)}{(A-1)} \Rightarrow \frac{B^2-1}{B-1} > \frac{A^2-1}{A-1}$$

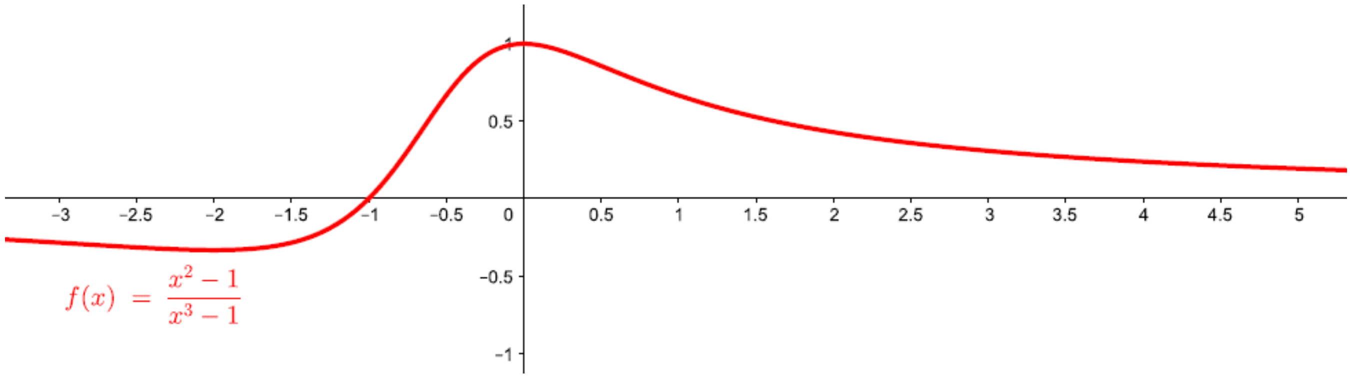
And since: $A-1>0$ and $B^2-1>0$ we have:

$$\frac{A-1}{B-1} > \frac{A^2-1}{B^2-1}$$

Finally, let's consider

$$y = \frac{x^2-1}{x^3-1}$$

which is decreasing in $]0,\ +\infty[$



Since $B > A > 1 > 0$

$$\frac{B^2-1}{B^3-1} < \frac{A^2-1}{A^3-1} \xRightarrow[A^2-1>0;\, B^3-1>0]{} \frac{B^2-1}{A^2-1} < \frac{B^3-1}{A^3-1} \Rightarrow \frac{A^2-1}{B^2-1} > \frac{A^3-1}{B^3-1}$$

In short:

$$\frac{A+1}{B+1} > \frac{A-1}{B-1} > \frac{A^2-1}{B^2-1} > \frac{A^3-1}{B^3-1}$$

November 18: Find the pairs of prime numbers (x, y) such that x + y and x – y are also primes. Is the sum of the four also prime?

Solution: If x and y are both prime and odd their sum will be even. Therefore, if we are looking for x and y primes with x + y prime, x or y must be even prime, that is, 2. Suppose that y = 2. We then look for x prime with x – 2 and x + 2 primes. But between x – 2, x and x + 2 there is a multiple of 3 (If x = 0 (3) is already proven, if x = 1 (3) then x + 2 = 0(3) and if x = 2 (3) then x – 2 = 0(3)). The only possibility is x – 2 = 3 (the only multiple of three that is prime). Therefore, x = 5 is the only possibility.

There is only one pair of primes (2, 5) so their sum and difference (3 and 7) are prime. The sum of the four is (2 + 3 + 5 + 7 =) 17 is also prime.

November 20: In a circle of radius $r = 5/\sqrt{2}$ we inscribe a right triangle with legs natural numbers. Find its perimeter

Solution: If it is a right triangle, the hypotenuse must be a diameter, (because then, the inscribed angle is 90° and the central angle is 180°). Therefore, if a and b are the legs of the right triangle, it must be true:

$$a^2 + b^2 = \left(\frac{2 \cdot 5}{\sqrt{2}}\right)^2 = 50$$

And since a and b must be natural

a	$50 - a^2$	$¿ b = \sqrt{50 - a^2} \in \mathbb{N}?$
1	49	if (b = 1)
2	46	no
3	41	no
4	34	no
5	25	if (b = 5)
6	14	no
7	49	if (b = 7)

Therefore, the only admissible solutions are a = 1, b = 7 and a = b = 5. Therefore, the perimeter can be equal to $8 + 5\sqrt{2}$ or $10 + 5\sqrt{2}$.

<u>November 21:</u> How many integers between 3 and 89 cannot be written as the sum of exactly two numbers from the set {1, 2, 3, 5, 8, 13, 21, 34, 55}?

<u>Solution:</u> First we must realize that the sum of any two of the given numbers will not give the same result because each number, of the given numbers, is the sum of the previous two (they give us the first terms of the Fibonacci sequence).

Let's calculate how many numbers we can get by adding two of the dice. By the previous point there will be as many numbers as combinations of nine elements taken two by two, that is:

$$C_2^9 = \binom{9}{2} = \frac{9!}{2! \cdot 7!} = 36$$

Since the smallest number that we can obtain by adding two of those provided is (1 + 2 =) 3 and the largest is (34 + 55 =) 89, there are (89 – 2 =) 87 candidates to be obtained. Since we can get 36, we can't get (87 – 36 =) 51.

<u>November 23-30:</u> The number of years I turned yesterday is a two-digit prime. If I add my son's age, I get another prime number. But if the remainder I get a multiple of 3 and 11. If I add the digits of my age and the digits of my son's age, I get eight. What are the ages?

<u>Solution:</u> Let 10a + b and 10c + d be the ages of mother and child, respectively. We have, then, of the conditions of the statement:

A. a> c (with c possibly zero) with a, b, c, d $\in \mathbb{N}$
B. 10a + b is prime
C. 10(a + c) + (b +d) is prime
D. 10 (a – c) + (b – d) is multiple of 33
E. a + b + c + d = 8 ($\Rightarrow$ a $\leq$ 8)

If a = 8, from condition E, we have that b = c = d = 0, so that; 10a + b = 80 which is not prime, which contradicts B

If a = 7, 2-digit primes with tens equal to 7 are: 71, 73 and 79.

If b = 1, we have, by E, that c = d = 0, which contradicts D.

If b = 3 or b = 9, E is contradicted

If a = 6, 2-digit primes with tens equal to 6 are: 61 and 67.

$$\left.\begin{array}{c}\text{If } b = 1 \text{ (by E)}\\ c + d = 2\end{array}\right\} \Rightarrow \begin{cases}\text{if } c = 2 \text{ and } d = 0 & \Rightarrow 61 - 20 = 41 \neq \widehat{\dot{33}}\\ \text{if } c = 1 \text{ and } d = 1 & \Rightarrow 61 - 11 = 50 \neq \widehat{\dot{33}}\\ \text{if } c = 0 \text{ and } d = 2 & \Rightarrow 61 - 02 = 59 \neq \widehat{\dot{33}}\end{cases}$$

If b = 7 (for E) we have that c + d = -5, which contradicts A

If a = 5, 2-digit primes with tens equal to 5 are: 53 and 59.

If b = 3, we have, by E, that c = d = 0, which contradicts D.

If b = 9, (by E) we have, that c + d = - 6, which contradicts A

If a = 4, 2-digit primes with tens equal to 4 are: 41, 43 and 49.

$$\left.\begin{array}{c}\text{If } b = 1 \text{ (by E)}\\ c + d = 3\end{array}\right\} \Rightarrow \begin{cases}\text{if } c = 3 \text{ and } d = 0 & \Rightarrow 41 - 30 = 11 \neq \widehat{\dot{33}}\\ \text{if } c = 2 \text{ and } d = 1 & \Rightarrow 41 - 21 = 20 \neq \widehat{\dot{33}}\\ \text{if } c = 1 \text{ and } d = 2 & \Rightarrow 41 - 12 = 29 \neq \widehat{\dot{33}}\\ \text{if } c = 0 \text{ and } d = 3 & \Rightarrow 41 - 03 = 38 \neq \widehat{\dot{33}}\end{cases}$$

$$\left.\begin{array}{c}\text{If } b = 3 \text{ (by E)}\\ c + d = 1\end{array}\right\} \Rightarrow \begin{cases}\text{if } c = 1 \text{ and } d = 0 & \Rightarrow \mathbf{41 \text{ and } 10} \text{ ¡}\mathbf{SOLUTION}\text{!}\\ \text{if } c = 0 \text{ and } d = 1 & \Rightarrow 41 - 01 = 40 \neq \widehat{\dot{33}}\end{cases}$$

If b = 9, (by E) we have, that c + d = - 5, which contradicts A

If a = 3, 2-digit primes with tens equal to 3 are: 31 and 37

$$\left.\begin{array}{c}\text{if } b = 1 \text{ (by E)}\\ c + d = 4 \text{ (by A)}\end{array}\right\} \Rightarrow \begin{cases}\text{if } c = 2 \text{ and } d = 2 & \Rightarrow 31 - 22 = 9 \neq \widehat{\dot{33}}\\ \text{if } c = 1 \text{ and } d = 3 & \Rightarrow 31 - 13 = 18 \neq \widehat{\dot{33}}\\ \text{if } c = 0 \text{ and } d = 3 & \Rightarrow 31 - 03 = 28 \neq \widehat{\dot{33}}\end{cases}$$

If b = 7, (by E) we have that c + d = - 2, which contradicts A

Values for a less than or equal to 2 are not a solution because an age of the mother of twenty-something minus the age of the child cannot be 33

<u>November 24-25:</u> Dani randomly chooses three numbers from the set {1, 2, 3, 4} and Laia one from the set {2, 4, 6, 8, 10}. What is the probability that the number drawn by Laia is greater than the sum of those extracted by Dani?

<u>Solution:</u> For Dani's extractions: Picking 3 numbers from {1, 2, 3, 4} and adding them is equivalent to extracting only 1 and adding the undrawn numbers. Then the possible outcomes and their probabilities for Dani are:

$$\text{Dani} \Rightarrow \begin{cases}1 \Rightarrow S_D = 9 \longrightarrow \dfrac{1}{4}\\ 2 \Rightarrow S_D = 8 \longrightarrow \dfrac{1}{4}\\ 3 \Rightarrow S_D = 7 \longrightarrow \dfrac{1}{4}\\ 4 \Rightarrow S_D = 6 \longrightarrow \dfrac{1}{4}\end{cases}$$

Whereas, for Laia, we have:

$$\text{Laia} \Rightarrow \begin{cases} 2 \Rightarrow S_L = 2 \longrightarrow \frac{1}{5} \\ 4 \Rightarrow S_L = 4 \longrightarrow \frac{1}{5} \\ 6 \Rightarrow S_L = 6 \longrightarrow \frac{1}{5} \\ 8 \Rightarrow S_L = 8 \longrightarrow \frac{1}{5} \\ 10 \Rightarrow S_L = 10 \longrightarrow \frac{1}{5} \end{cases}$$

$$\begin{aligned} P(L > D) &= P\Big((S_L = 8) \cap \big((S_D = 7) \cup (S_D = 8)\big)\Big) + P\big((S_L = 10) \cap (S_D \text{ cualquiera})\big) \\ &= \frac{1}{5} \cdot \left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{5} = \frac{3}{10} \end{aligned}$$

<u>November 26:</u> What is the probability that a 10-digit number contains all 10 digits?

<u>Solution:</u> Let's first calculate the ten-digit numbers that we can generate. Taking into account that they start with 0 they only have nine digits, we have:

Possible cases: $VR_{10}^{10} - VR_{10}^{9} = 10^{10} - 10^{9}$

Favourable cases will be the 10-digit numbers that contain all ten digits. Namely:

Favourable cases: $V_{10}^{10} - V_{9}^{9} = 10! - 9!$

The requested probability is then:

$$P = \frac{10! - 9!}{10^{10} - 10^{9}} = \frac{9! \cdot (10 - 1)}{10^{9} \cdot (10 - 1)} = \frac{9!}{10^{9}}$$

<u>November 27-28:</u> The mean of three consecutive odd numbers is 7. If we add another positive integer m, different from the three, the mean of the four is another integer. Find the three smallest values of m.

<u>Solution:</u> Let a, a + 2, a + 4 be the consecutive odd numbers in the sentence. We will have:

$$\frac{a + a + 2 + a + 4}{3} = \frac{3a + 6}{3} = 7 \Rightarrow a = 5$$

Then the three consecutive odd numbers in the statement are: 5, 7 and 9. We now demand that the mean of the four natural numbers be another natural:

$$\frac{5 + 7 + 9 + m}{4} = \frac{21 + m}{4} = 5 + \frac{1 + m}{4} \in \mathbb{N} \Leftrightarrow m = 3(4) = \{3, 7, 11, 15, \cdots\}$$

Since m is different from the initial three odd numbers, we have that the answer is m = 3, 11 and 15