SOLUTIONS OCTOBER 2021

PROBLEMS OF THE CANADIAN MATHEMATICAL OLYMPIAD: 1974, 1975, 1976. OME PREPARATION. (http://cms.math.ca/Competitions/CMO/) ORGANIZATION AND SOLUTIONS: Rafael Martínez Calafat. Retired teacher.

October 1-2: (1).- If $\mathbf{x}=\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{n}}\right)^{\mathbf{n}}$ and $\mathbf{y}=\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{n}}\right)^{\mathbf{n+1}}$ to compare $\mathbf{x^y}$ which $\mathbf{y^x}$. (2).- Prove that, for every positive integer n:

$$\mathbf{1^2-2^2+3^2-4^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}\cdot n^2=(-1)^{n+1}\cdot(1+2+\cdots+n)}$$

Solution: For (1), we have: For some particular values of n we obtain that $x^y = y^x$. So we try to prove the equality of the two expressions for any value of n.

$$x^y=\left[\left(1+\frac{1}{n}\right)^n\right]^{\left(1+\frac{1}{n}\right)^{n+1}}=\left(\frac{n+1}{n}\right)^{n\cdot\left(\frac{n+1}{n}\right)^{n+1}}=\left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}}$$

$$y^x=\left[\left(1+\frac{1}{n}\right)^{n+1}\right]^{\left(1+\frac{1}{n}\right)^{n}}=\left(\frac{n+1}{n}\right)^{(n+1)\cdot\left(\frac{n+1}{n}\right)^{n}}=\left(\frac{n+1}{n}\right)^{\frac{(n+1)^{n+1}}{n^n}}$$

the two expressions coincide, since the same base raised to the same exponent is obtained.

For (2) we will use induction on n. In more detail, we will prove that the formula is true for n = 1 and n = 2, we will assume that the formula is true for n – 1, and we will prove it for n + 1.

For n = 1, we have:

$$1^2=1=(-1)^{1+1}\cdot 1$$

For n = 2, we have:

$$1^2-2^2=1-4=-3=(-1)\cdot 3=(-1)^{2+1}\cdot(1+2)$$

Suppose the formula is true for n – 1:

$$1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2=(-1)^n\cdot\big(1+2+3+\cdots+(n-1)\big)$$

and let's see it for n + 1:

$$1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}n^2+(-1)^{n+2}(n+1)^2=$$

$$=1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}n^2+(-1)^{n+2}(n^2+2n+1)=$$

$$=1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}n^2+(-1)^{n+1}(-n^2-2n-1)=$$

$$1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}n^2-(-1)^{n+1}n^2+(-1)^{n+1}(-2n-1)=$$

$$=1^2-2^2+3^2+\cdots+(-1)^n\cdot(n-1)^2+(-1)^{n+1}(-2n-1)=\left\{\begin{matrix}\text{hypothesis of}\\ \text{induction}\end{matrix}\right\}=$$

$$=(-1)^n\big(1+2+3+\cdots+(n-1)\big)+(-1)^n(2n+1)=$$

$$=(-1)^n\big(1+2+3+\cdots+(n-1)+n+(n+1)\big)=(-1)^{n+2}\big(1+2+3+\cdots+n+(n+1)\big)$$

October 4-5: Let n be a fixed positive integer. To any choice of n real numbers satisfying $0 \le x_i \le 1$ ($i \in \{1, 2, .. ,n\}$) we associate the sum:

$$(*)_n = \sum_{1 \le i < j \le n} |x_i - x_j| = |x_1 - x_2| + |x_1 - x_3| + \cdots + |x_1 - x_n| + |x_2 - x_3| + .. + |x_2 - x_n| + \ldots + |x_{n-1} - x_n|$$

Find the largest possible value of this sum.

Solution: Note that $(*)_n$ is a kind of measure of dispersion of the real numbers $\{x_i\}_{i=1}^{n}$. Let's see how to get that maximum sum. If we have two points, the maximum sum is reached with:

$$x_1 = 0; \;\; x_2 = 1; \quad (*)_2 = |x_1 - x_2| = 1$$

If we have three points, the maximum sum is reached with:

$$x_1 = 0; \;\; x_2 = 1; \; x_3 = 0 \quad (*)_3 = |x_1 - x_2| + |x_1 - x_3| + (*)_2 = 1 + 0 + 1 = 2$$

If we have four points, the maximum sum is reached with:

$$x_1 = 0; \;\; x_2 = 1; \; x_3 = 0; \; x_4 = 1 \quad (*)_4 = |x_1 - x_2| + |x_1 - x_3| + |x_1 - x_4| + (*)_3 = 1 + 0 + 1 + 2 = 4$$

If we have five points, the maximum sum is reached with:

$$x_1 = 0; \; x_2 = 1; \; x_3 = 0; \; x_4 = 1; \; x_5 = 0 \quad (*)_5 = |x_1 - x_2| + |x_1 - x_3| + |x_1 - x_4| + |x_1 - x_5| + (*)_4 = 1 + 0 + 1 + 0 + 4 = 6$$

It is observed that:

$$(*)_n = |x_1 - x_2| + |x_1 - x_3| + \cdots + |x_1 - x_n| + (*)_{n-1}; \quad (*)_2 = 1; \; (*)_3 = 2; \; (*)_4 = 4; \; (*)_5 = 6; \; \cdots$$

And how:

$$|x_1 - x_2| + |x_1 - x_3| + \cdots + |x_1 - x_n| = \left[\frac{n}{2}\right]$$

where $[\,\cdot]$ is the integer part function, we have:

$$(*)_n = \left[\frac{n}{2}\right] + (*)_{n-1}; \quad \text{con } (*)_2 = 1$$

October 6: Simplify:

$$\sqrt[3]{\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \cdots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n}}$$

Solution: We have:

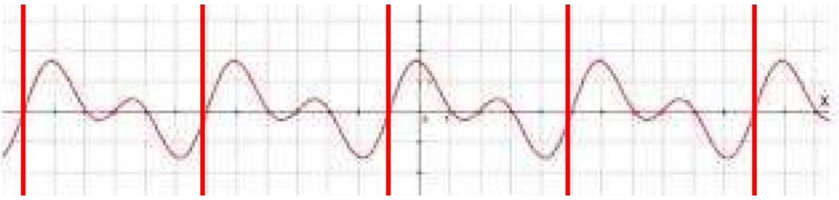
$$\sqrt[3]{\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \cdots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n}} = \left(\frac{1 \cdot 2 \cdot 4 + 2 \cdot 4 \cdot 8 + \cdots + n \cdot 2n \cdot 4n}{1 \cdot 3 \cdot 9 + 2 \cdot 6 \cdot 18 + \cdots + n \cdot 3n \cdot 9n}\right)^{1/3}$$
$$= \left\{\begin{matrix} \text{each addend of the numerator} \\ \text{contains the factor } 2^3 \\ \text{each sum of the denominator} \\ \text{contains the factor } 3^3 \end{matrix}\right\} = \left(\frac{2^3(1 + 2^3 + 3^3 + \cdots + n^3)}{3^3(1 + 2^3 + 3^3 + \cdots + n^3)}\right)^{1/3} = \left(\frac{2^3}{3^3}\right)^{1/3} = \frac{2}{3}$$

October 7-8: A function y = f(x) is said to be periodic if there exists a positive real number p such that f(x + p) = f(x) for all x. For example, y = sin x is periodic of period 2π. Is the function periodic:

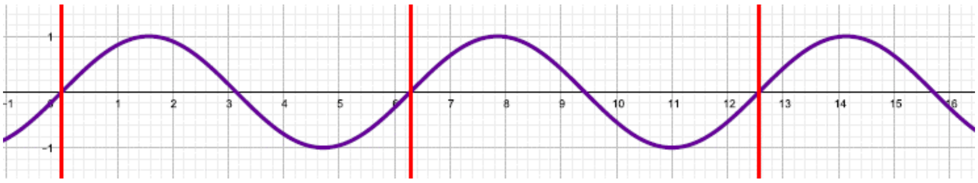
y = sen (x^2)?

Prove your claim.

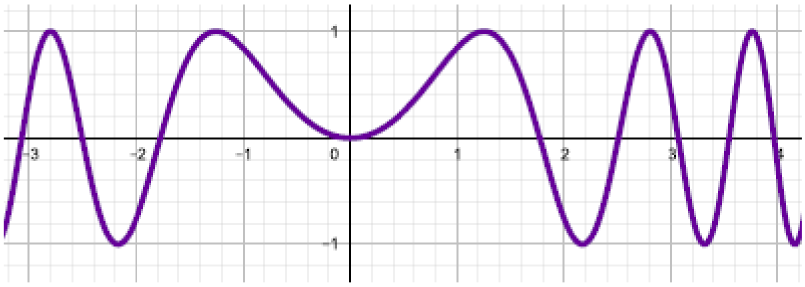
Solution: If a function is periodic, there is a part of its graph that repeats indefinitely both to the right and to the left.:



The graph of y = sin x is periodic, of period 2π



Since x^2 gets very large as x gets large, y = sin x^2 will mean that the sine wave will get tighter as we move to the right or to the left (apart from the fact that the graph would be that of an even function). That is, the graph of the function y = sin x^2 will be of the type:



Which makes it impossible for it to be a periodic function. To prove it we will assume that it is periodic and we will arrive at an absurdity. Let p be the period of y = sin x^2. Then:

$$\mathrm{sen}(x+p)^2 = \mathrm{sen}x^2 \quad \forall x$$

Giving x the value 0:

$$\mathrm{sen}p^2 = \mathrm{sen}0 = 0 \quad \Rightarrow \quad p^2 = k\pi \ \text{ for some } k \in \mathbb{N} \quad \Rightarrow \quad p = \sqrt{k\pi} \ \text{ for some } \ k \in \mathbb{N}$$

we will have then:

$$\mathrm{sen}\left(x+\sqrt{k\pi}\right)^2 = \mathrm{sen}x^2 \quad \forall x$$

and now we look for some value of x for which the left part of the previous equal to different from the right part.

we will have, doing $x = \sqrt{\frac{k\pi}{2}}$

$$\mathrm{sen}\left(\sqrt{\frac{k\pi}{2}}+\sqrt{k\pi}\right)^2 = \begin{cases} = \mathrm{sen}\left[\left(1+\sqrt{2}\right)^2 \dfrac{k\pi}{2}\right] \\ = \mathrm{sen}\left(\dfrac{k\pi}{2}\right) = \begin{cases} 1 \text{ si } k = 1,3,5,\cdots \\ 0 \text{ si } k = 2,4,6,\cdots \end{cases} \end{cases}$$

but:

$$5 < \left(1+\sqrt{2}\right)^2 < 6$$

from where:

$$5\frac{k\pi}{2} < \left(1+\sqrt{2}\right)^2\frac{k\pi}{2} < 3k\pi$$

so that:

$$\mathrm{sen}\left[\left(1+\sqrt{2}\right)^2\frac{k\pi}{2}\right] \neq \begin{cases}\mathrm{sen}\frac{5k\pi}{2} = \begin{cases}1 \text{ si } k = 1,3,5,\cdots \\ 0 \text{ si } k = 2,4,6,\cdots\end{cases} \\ \mathrm{sen}3k\pi = 0\end{cases}$$

that contradicts

$$\mathrm{sen}\left[\left(1+\sqrt{2}\right)^2\frac{k\pi}{2}\right] = \begin{cases}1 \\ 0\end{cases}$$

October 9: A succession $a_1, a_2, a_3, \cdots\cdots$ meets that

(1) $a_1 = ½$
(2) $a_1 + a_2 + \cdot\cdot + a_n = n^2 \cdot a_n$, for any n.

Determine the value of a_n.

Solution: We have:

$$\begin{aligned} a_1 + a_2 + \cdots + a_{n-1} + a_n &= n^2 a_n \\ a_1 + a_2 + \cdots + a_{n-1} &= (n-1)^2 a_{n-1} \end{aligned}$$

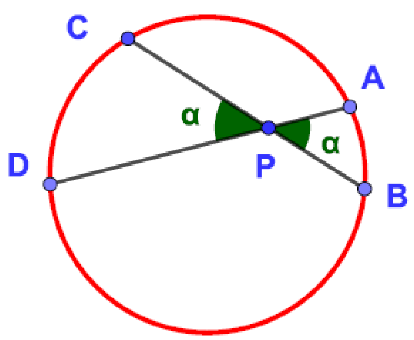
and subtracting both expressions:

$$a_n(n^2-1) = (n-1)^2 a_{n-1} \overset{n\neq 1}{\Rightarrow} a_n = \frac{n-1}{n+1}a_{n-1}$$

from where:

$$\begin{aligned} a_n &= \frac{n-1}{n+1}a_{n-1} = \frac{n-1}{n+1}\cdot\frac{n-2}{n}a_{n-2} = \frac{n-1}{n+1}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n-1}a_{n-3} = \cdots = \frac{n-1}{n+1}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\cdots\cdot\frac{1}{3}a_1 \\ &= \frac{n-1}{n+1}\cdot\frac{n-2}{n}\cdot\frac{n-3}{n-1}\cdot\cdots\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{(n-1)!}{(n+1)!} = \frac{1}{n(n+1)} \end{aligned}$$

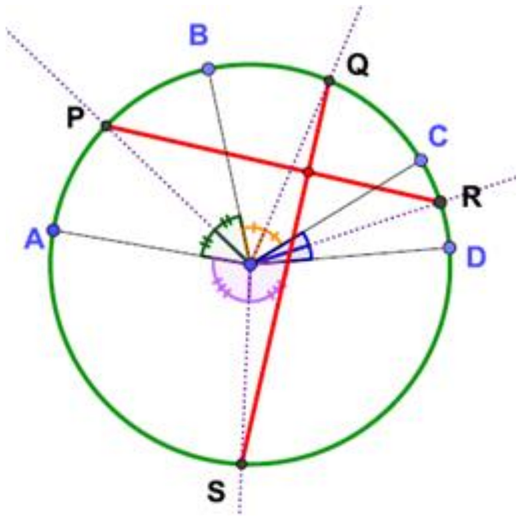
October 11: Let A, B, C and D be four consecutive points on a circle and P, Q, R and S be the midpoints of the arcs AB, BC, CD and DA. prove that PR ⊥ QS.



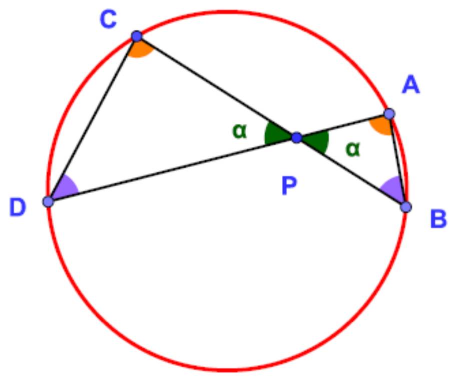
Previous lemma: If P is an interior point of the circle, the interior angle is defined as the angle ∠APB = ∠CPD.

In the context of the above definition:

$$\alpha = \frac{\widehat{AB} + \widehat{DC}}{2}$$



Motto Proof:

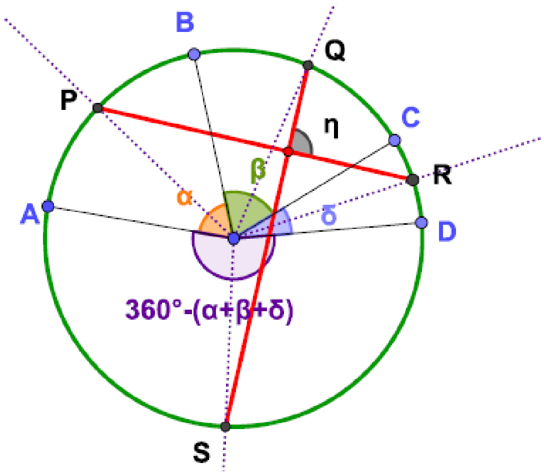


Let us consider the triangle ΔPCD. In it we have:

$$\alpha + \angle D + \angle C = 180° \;\Rightarrow\; \alpha + \frac{\overset{\frown}{CA}}{2} + \frac{\overset{\frown}{BD}}{2} = 180°$$

$$2\alpha + \overset{\frown}{CA} + \overset{\frown}{BD} = 360° = \overset{\frown}{CA} + \overset{\frown}{AB} + \overset{\frown}{BD} + \overset{\frown}{DC} \;\Rightarrow\; \alpha = \frac{\overset{\frown}{AB} + \overset{\frown}{DC}}{2}$$

Solution:



We must prove that η = 90°. From the previous lemma:

$$\eta = \frac{\overset{\frown}{QR} + \overset{\frown}{SP}}{2} = \frac{\overset{\frown}{QC} + \overset{\frown}{CR} + \overset{\frown}{SA} + \overset{\frown}{AP}}{2}$$

$$= \frac{\frac{\beta}{2} + \frac{\delta}{2} + \frac{360° - (\alpha + \beta + \delta)}{2} + \frac{\alpha}{2}}{2} = \frac{180°}{2} = 90°$$

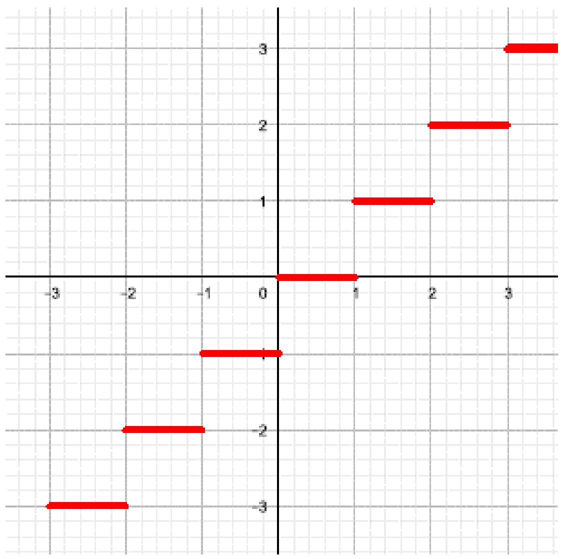
October 12-13: For each real r, is defined:

[r] = max {z∈ℤ | z ≤ r}

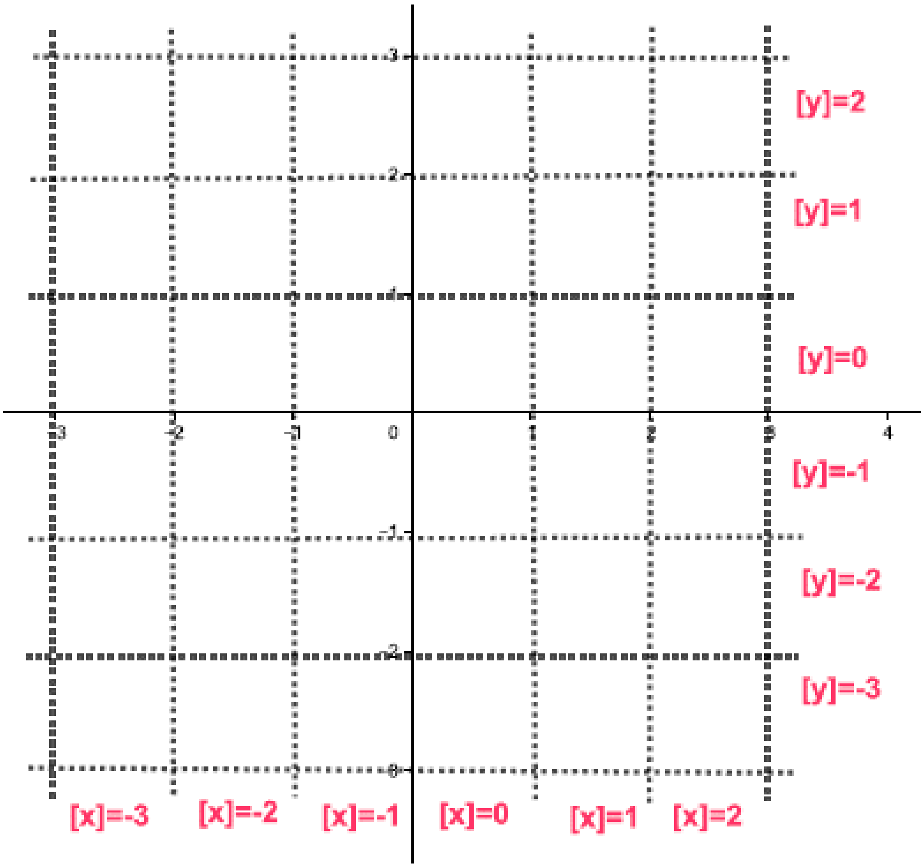
(e. g. [6] = 6; [π] = 3; [-1,5] = -2)). Plot in the (x, y) plane the set of points:

$[x]^2 + [y]^2 = 4$

Solution: From the definition of [r] we have that its graph is:

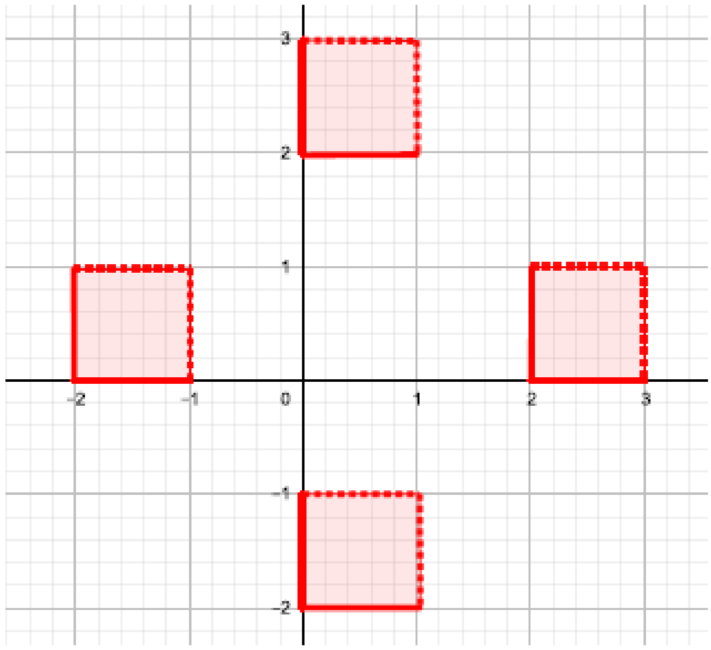


Therefore:



With this, the x-y plane is partitioned into squares. In each square we have calculated $[x]^2 + [y]^2$

Therefore, what is requested in the statement is:



<u>October 14:</u> Given four weights that are in arithmetic progression and a two-beam balance, show how to find the largest weight using the balance only twice.

<u>Solution:</u> To be:

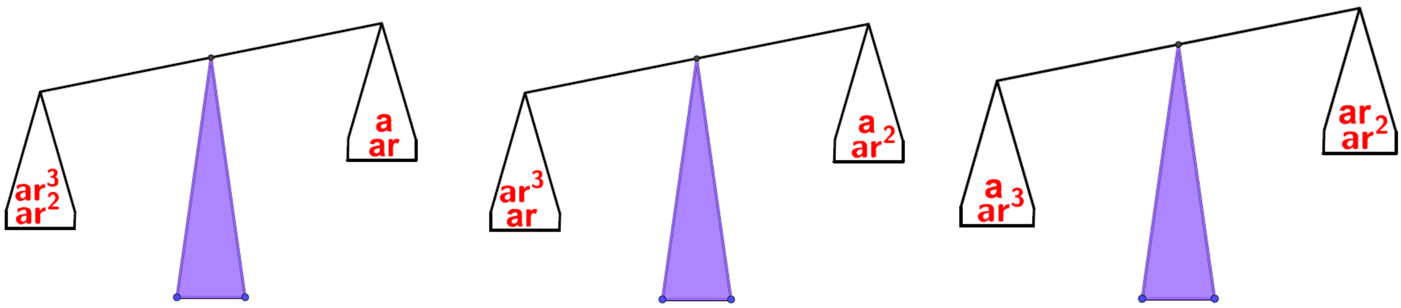
$$a; \;\; ar; \;\; ar^2; \;\; ar^3$$

the weights with r > 1 (and so the weights are growing).

Given the four weights, we separate them into two groups of two weights each.

A. We put one pair of weights on one pan and the other pair of weights on the other pan. The pair of weights that contains the weight with r raised to 3 is the heavier of the two groups.
B. Locating the pair that contains the weight with r raised to the cube, we put each of the two weights on the pans of the balance. The largest weight is the one on the lowest pan.

We will prove, now, the affirmation exposed in A: When making the two groups of two weights, there are three possibilities:



We have to prove that r > 1:

$$A-1: \;\; r^2 + r^3 > 1 + r$$
$$A-2: \;\; r^3 + r > \; r^2 + 1$$
$$A-3: \;\; r^3 + 1 > \; r^2 + r$$

We have, for A-1:

$$r > 1 \;\Rightarrow \begin{Bmatrix} (*)\; r^2 > r \\ (**)\;\; r^3 > 1 \end{Bmatrix} \;\Rightarrow\; \mathbf{r^3 + r^2 > r + 1}$$

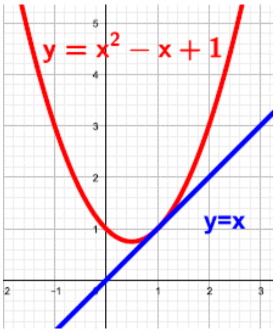
(*) Multiplying each member of the inequality r > 1, by r (>0)

(**) Multiplying by r^2 (>0) the inequality r > 1: $r^3 > r^2 > 1$

For A-2: Multiplying by r^2+1 (>0) the inequality r > 1:

$$r(r^2+1) = \mathbf{r^3 + r > r^2 + 1}$$

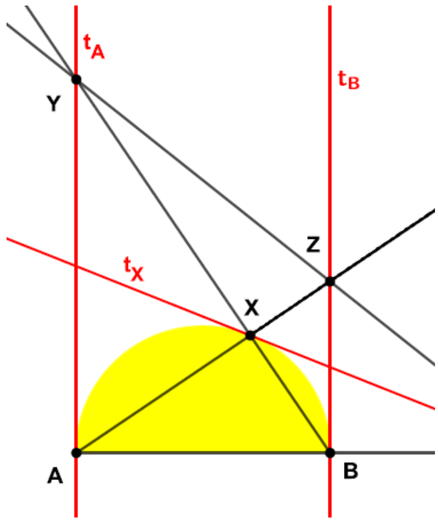
For A-3:



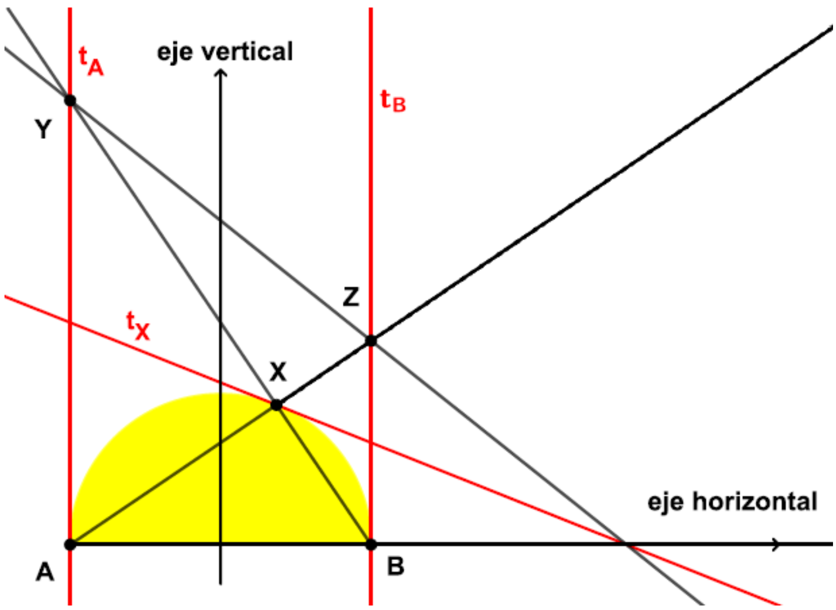
From the attached graph, we have, (which for r > 1), $r^2 - r + 1 > r$. Multiplying each member of this inequality by r + 1 (>0), we have:

$$(r+1)\cdot(r^2-r+1) = \mathbf{r^3 + 1 > r^2 + r} = (r+1)\cdot r$$

October 15-16: Given a circle of diameter AB and a point X different from A and B, let t_A, t_B and t_X be the tangents to the circle at A, B and X. Let Z be the intersection of AX with t_B and Y be the intersection of BX with t_A . Prove that the three lines AB, t_X and ZY are concurrent or parallel..



Solution:



We choose the reference system so that the horizontal axis is the line AB and the vertical axis is parallel to t_A through the midpoint of the segment AB. In this case, we have:

$$A(-a,0);\quad B(a,0);\quad x^2+y^2=a^2$$

$$X(\alpha,\beta)\;\Rightarrow\;\alpha^2+\beta^2=a^2\;(-a\neq\alpha\neq a)$$

$$t_A \equiv x=-a;\quad t_B\equiv x=a$$

t_X:

$$x^2 + y^2 = a^2 \;\Rightarrow\; 2x + 2yy' = 0 \;\Rightarrow y' = -\frac{x}{y} \;\Rightarrow\; y'|_{(\alpha,\beta)} = -\frac{\alpha}{\beta}$$

$$t_X \equiv y - y_0 = -\frac{\alpha}{\beta}(x - x_0) \;\equiv\; \mathbf{y = -\frac{\alpha}{\beta}(x - \alpha) + \beta}$$

AX

$$\left.\begin{matrix} A(-a,0) \\ X(\alpha,\beta) \end{matrix}\right\} \;\Rightarrow\; \frac{y-0}{x+a} = \frac{\beta-0}{\alpha+a} \;\Rightarrow\; \mathbf{y = \frac{\beta}{\alpha+a}(x+a)}$$

$Z = t_B \cap AX$

$$AX \equiv\; y = \frac{\beta}{\alpha+a}(x+a) \equiv \{t_B \equiv x = a\}\, y = \frac{2\beta a}{\alpha+a} \;\Rightarrow\; \mathbf{Z\left(a, \frac{2\beta a}{\alpha+a}\right)}$$

BX

$$\left.\begin{matrix} B(a,0) \\ X(\alpha,\beta) \end{matrix}\right\} \;\Rightarrow\; \frac{y-0}{x-a} = \frac{\beta-0}{\alpha-a} \;\Rightarrow\; \mathbf{y = \frac{\beta}{\alpha-a}(x-a)}$$

$Y = t_X \cap BX$

$$BX \equiv\; y = \frac{\beta}{\alpha-a}(x-a) \equiv \{t_A \equiv x = -a\}\, y = \frac{-2\beta a}{\alpha+a} \;\Rightarrow\; \mathbf{Y\left(-a, \frac{2\beta a}{a-\alpha}\right)}$$

YZ

$$\left.\begin{matrix} Z\left(a, \dfrac{2\beta a}{\alpha+a}\right) \\ Y\left(-a, \dfrac{2\beta a}{a-\alpha}\right) \end{matrix}\right\} \;\Rightarrow\; \frac{y - \frac{2\beta a}{\alpha+a}}{x-a} = \frac{\frac{2\beta a}{a-\alpha} - \frac{2\beta a}{\alpha+a}}{-2a} \;\Rightarrow\; \mathbf{y = \frac{2\beta\alpha}{\alpha^2 - a^2}(x-a) + \frac{2\beta a}{\alpha+a}}$$

$C = YZ \cap t_X$

$$YZ \equiv y = \frac{2\beta\alpha}{\alpha^2 - a^2}(x-a) + \frac{2\beta a}{\alpha+a}\;,\quad t_X \equiv y = -\frac{\alpha}{\beta}(x-\alpha) + \beta$$

$$\frac{2\beta\alpha}{\alpha^2 - a^2}(x-a) + \frac{2\beta a}{\alpha+a} = -\frac{\alpha}{\beta}(x-\alpha) + \beta \cdots (\text{recordar } \alpha^2 + \beta^2 = a^2) \cdots \Rightarrow x = \frac{a^2}{\alpha} \;(\alpha \neq 0)$$

$$y = -\frac{\alpha}{\beta}\left(\frac{a^2}{\alpha} - \alpha\right) + \beta = \cdots = 0 \;\Rightarrow\; \mathbf{t_X \cap AB \cap YZ = C\left(\frac{a^2}{\alpha}, 0\right) \;(\alpha \neq 0)}$$

For the case $\alpha = 0$ we have:

$$\mathbf{t_X \equiv y = \beta; \quad YZ \equiv\; y = \frac{2\beta a}{a} = 2\beta}$$

what are parallel lines

<u>October 18-19:</u> We have an unlimited supply of 8 cent and 15 cent stamps. Some postage amounts cannot be obtained exactly, e. g. 7 cents or 29 cents. What is the largest quantity that cannot be obtained exactly, i. e. the amount of postage that is not exactly achievable, while all larger amounts are achievable?

<u>Solution:</u> Let's imagine the following table:

x\y	0	1	2	3	4	5	....
0	0	15	30	45	60	75	
1	8	23	38	53	68	83	
2							
3							
4							
5							
:							

In it, in each cell, is the result of performing the operation:

$$8 \cdot (\text{starting number in the row}) + 1 \cdot (\text{starting number in the column})$$

The table, prepared in this way, contains the values of n for which the equation 8x + 15y = n has a solution, that is, the exact postages that can be generated with 8-cent and 15-cent stamps. The table has the following properties:

- If a number n is in a cell, in the cell immediately below it is the number n + 8 and the cell to the left of the first contains the number n + 15, since

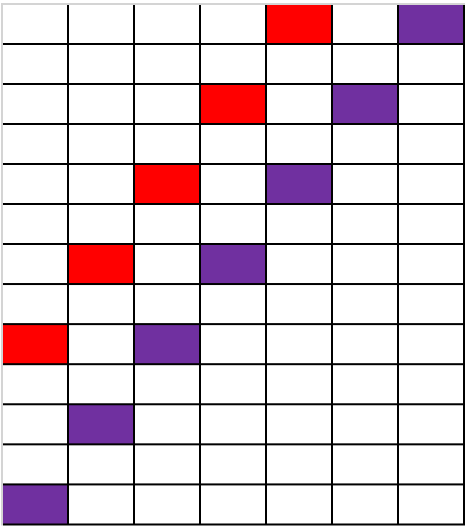
$$n = 8x + 15y \;\Rightarrow\; \begin{cases} n + 8 = 8(x + 1) + 15y \\ n + 15 = 8x + 15(y + 1) \end{cases}$$

- Since 15=2 8 – 1 (=16 – 1) we will have that, if n is in a cell, one cell to the left and two cells below it is in number n + 1 and so on

		n	n+15
		n+8	
	n+1		
n+2			

$$n = 8x + 15y \Rightarrow\; n + 1 = 8(x + 2) + 15(y - 1) \Rightarrow n + 2 = 8(x + 4) + 15(y - 2)$$

- If we call coordinates of a cell to the position of the cell as in a matrix, we define diagonal associated to a number located in the first row (1,k) to the cells located in the table and in the positions: (1, k); (3, k-1); (5, k-2); ⋯ ; (2k-1, 1)



- Let us consider the cell of coordinates (1, k) (coefficient of 8 = 0, coefficient of 15 = k – 1) and we demand that it be in number n:

$$n = 15 \cdot (k - 1)$$

- We also require that all the cells of the associated diagonal be in the table and that the associated diagonal end in column 1. That is, we can go down 14 cells and move to the left k – 1 cells ending at position (2k – 1, k–1). In that cell will be the number n + 7 (k = 8 ⇒ coefficient of 8 = 14, coefficient of 15 = 0)

$$n + 7 = 14 \cdot 8 \;\;\Rightarrow\;\; n = 105$$

								(1,k)	
							n-15	n	
							n-7	n+8	
						n-14	n+1	.+16	
						n-6	n+9		
					n-13	n+2	.+17		
					n-5	n+10			
				n-12	n+3	.+18			
				n-4	.+11				
			n-11	n+4	.+19				
			n-3	.+12					
		n-10	n+5	.+20					
		n-2	.+13						
	n-9	n+6	.+21						
	n-1	.+14							
(2k-1, k-(k-1))=(15, 1) n-8	n+7	.+22							
	.+15								
	.+23								

The number n – 8 will not be in the table, since its coordinates fall outside the table. And also in the table will be all the numbers after n – 8 (= 105 – 8). That is, it will be missing in number 97 and there will be all those after 97. The answer to the statement is number 97.

Note: The values of n for which there is no solution in 8x + 15y = n can be obtained by filling in the described table and analysing the results

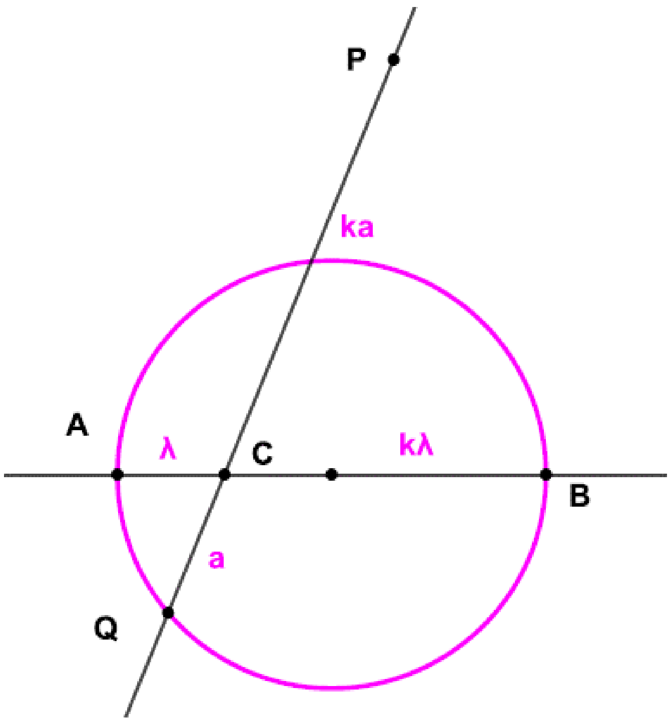
Valores de n sin solución = Franqueos imposibles de cumplir		0	1	2	3	4	5	6	7	8	9	10	11
1, 2, 3, 4, 5, 6, 7	0	0	15	30	45	60	75	90	105	120	135	150	165
9, 10 11, 12, 13, 14	1	8	23	38	53	68	83	98	113	128	143	158	173
17, 18, 19, 20, 21, 22	2	16	31	46	61	76	91	106	121	136	151	166	181
25, 26, 27, 28, 29	3	24	39	54	69	84	99	114	129	144	159	174	189
33, 34, 35, 36, 37	4	32	47	62	77	92	107	122	137	152	167	182	197
41, 42, 43, 44	5	40	55	70	85	100	115	130	145	160	175	190	205
49, 50, 51, 52	6	48	63	78	93	108	123	138	153	168	183	198	213
57, 58, 59	7	56	71	86	101	116	131	146	161	176	191	206	221
65, 66, 67	8	64	79	94	109	124	139	154	169	184	199	214	229
73, 74	9	72	87	102	117	132	147	162	177	192	207	222	237
81, 82	10	80	95	110	125	140	155	170	185	200	215	230	245
97	11	88	103	118	133	148	163	178	193	208	223	238	253
	12	96	111	126	141	156	171	186	201	216	231	246	261
	13	104	119	134	149	164	179	194	209	224	239	254	269
	14	112	127	142	157	172	187	202	217	232			

October 20-27: Let be given a circle and AB one of its diameters. Let C be a fixed point on AB and Q be a variable point on the circumference of the circle. Let P be a point on the line determined by Q and C for which:

$$ {}^{AC}/_{CB} = {}^{QC}/_{CP} $$

Describe the geometric locus of point P

Solution:



Let's draw the described situation.

Let k be the constant of proportionality between AC and CB:

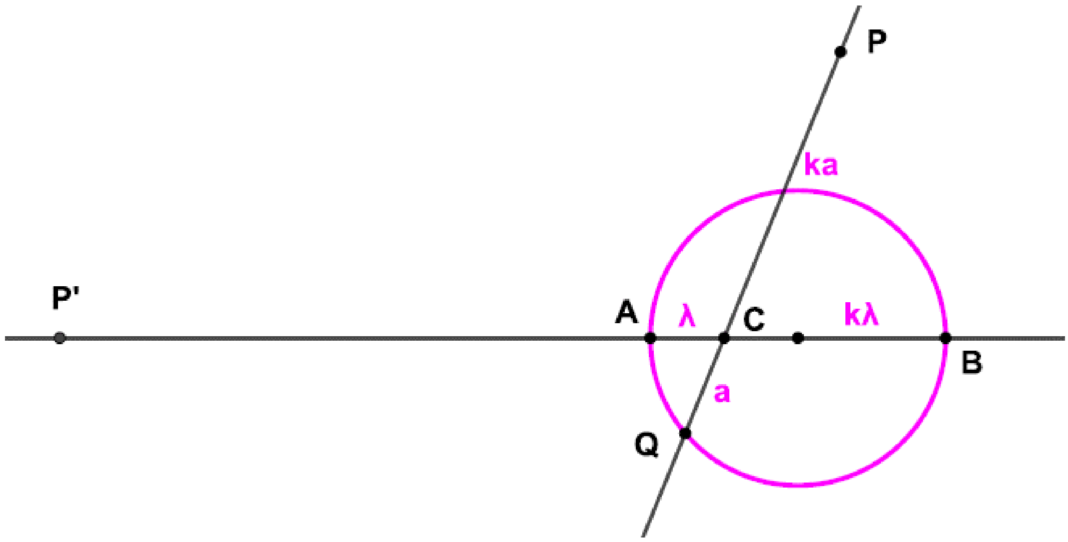
$$\frac{1}{k} = \frac{AC}{CB}\left(=\frac{\lambda}{\lambda k}\right) = \frac{QC}{CP}$$

We are asked for the locus of the points P by letting Q describe the circle of diameter AB. Since C is held fixed, if endpoint Q of segment QP describes a circle, endpoint P of segment QP also describes a circle. Furthermore, since C is in a diameter of the circle of diameter AB, a diameter of the circle drawn by P is also in the line that generates the segment AB. Making Q coincide with A and with B we will have the diameter of the circumference that describes P

If Q $\rightarrow$ A then P $\rightarrow$ B. Since B is a point on the circumference described by P

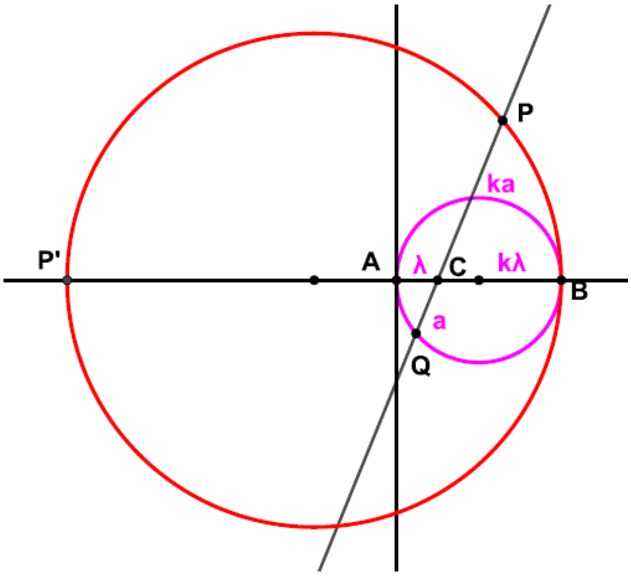
If Q $\rightarrow$ B then P $\rightarrow$ P' where P' is a point on the straight line passing through AB that satisfies:

$$CP' = k \cdot CB = k \cdot k\lambda = k^2\lambda$$



That is, if R is the radius of the circumference described by the point P we have:

$$2R = P'B = P'C + CB = k^2\lambda + k\lambda = \lambda k(k+1) \;\Rightarrow\; R = \frac{\lambda k(k+1)}{2}$$



October 21-22: 1.- Prove that if a number is rational, the decimal part, the integer part and the number cannot be in geometric progression.

2.- Find a positive number such that its decimal part, its integer part and the number are in geometric progression

Solution: For the first part we will proceed by reductio ad absurdum. Let e+p/q be the rational number and suppose that:

$$\frac{p}{q}, \qquad e, \qquad e+\frac{p}{q}$$

are in geometric progression. Then:

$$\frac{e}{\frac{p}{q}} = \frac{e+\frac{p}{q}}{e} \quad (= r)$$

simplify:

$$\frac{eq}{p} = \frac{eq+p}{qe} \;\Rightarrow\; e^2q^2 = (eq+p)p \;\Rightarrow\; e^2q^2 - epq - p^2 = 0$$

The last equation is a quadratic equation with respect to each of the unknowns: e, p and q, which must have integer solutions. For example, with respect to e, we have:

$$e = \frac{pq \pm \sqrt{p^2q^2 + 4p^2q^2}}{2q^2} = \begin{cases} \dfrac{(1+\sqrt{5})p}{2q} \\ \dfrac{(1-\sqrt{5})}{2q} \end{cases}$$

with what:

$$\frac{(1 \pm \sqrt{5})p}{2q} \in \mathbb{Z} \;\Rightarrow\; (1 \pm \sqrt{5}) \in \mathbb{Z} \;\Rightarrow\; \sqrt{5} \in \mathbb{Z}$$

which is absurd.

For the second part, we represent by {x} the decimal part and by [x] the integer part. If {x}, [x] and x are in geometric progression, we will have:

$$\frac{[x]}{\{x\}} = \frac{x}{[x]} \;\Rightarrow\; \frac{[x]}{x-[x]} = \frac{x}{[x]} \;\Rightarrow\; \frac{1}{\frac{x}{[x]}-1} = \frac{x}{[x]}$$

If we do

$$z = \frac{x}{[x]}$$

the above equation becomes:

$$\frac{1}{z-1} = z \;\Rightarrow\; z^2 - z - 1 = 0 \;\Rightarrow\; z = \frac{1 \pm \sqrt{5}}{2}$$

(we neglect the negative solution). Then if {x}, [x] and x are in geometric progression

$$\frac{x}{[x]} = \varphi = \frac{1+\sqrt{5}}{2} \quad (***)$$

We are going to prove that if {x}, [x] and x are in geometric progression then [x] = 1. By reductio ad absurdum.

If [x] = p ≥ 2, then: ({x}, p and x are in geometric progression)

$$\{x\} \cdot r = p \geq 2 \;\Rightarrow\; r \geq \frac{2}{\{x\}} > 2$$

If r > 2:

$$[x] \cdot r = x > 2[x] \quad \text{absurd!}$$

Finally, in (***) we have:

$$\frac{x}{[x]} = \frac{x}{1} = x = \varphi = \frac{1+\sqrt{5}}{2}$$

Note: The only positive number x such that {x}, [x] and x are in geometric progression is the golden ratio: φ

$$\{\varphi\} = \frac{1+\sqrt{5}}{2} - 1 = \frac{\sqrt{5}-1}{2}$$

$$r = \frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{\sqrt{5}+1}{2}$$

$$[\varphi] = 1 = \{\varphi\} \cdot r$$

$$\varphi = \frac{1+\sqrt{5}}{2} = [\varphi] \cdot r$$

October 23: Let n be a positive integer. Prove that if n is a power of 2 then n cannot be put as the sum of consecutive positive integers.

Solution: By reductio ad absurdum. Let's suppose $n = 2^p$ whith p ≥ 0. If p = 0, we have n = 1 and 1 cannot be expressed as the sum of at least two consecutive positive integers. If p = 1, we have n = 2 and 2 cannot be put as the sum of at least two positive integers. Be then p ≥ 2. Let's suppose 2^p is added from k (≥ 2) consecutive positive integers, then

$$2^p = (s+1) + (s+2) + \cdots + (s+k) = ks + (1+2+\cdots+k) = ks + \frac{k+1}{2}k = \frac{2ks + k^2 + k}{2} \; (*)$$

So:

$$0 = k^2 + k(2s+1) - 2^{p+1} \quad \text{con } p \geq 2 \text{ y } k \geq 2$$

is a quadratic equation (with respect to k) with at least one integer solution greater than or equal to 2. By the Cardano-Vietà relations, we have:

$$\text{sum of solutions} = -(2s+1)$$
$$\text{solutions product} = -2^{p+1}$$

That is, the sum of solutions is odd (therefore, one must be even and the other odd) and the product of solutions is a power of two and therefore the two solutions must be even (which contradicts that one is even and the other odd), since k = 1 is not a solution (if it were, we would have (*) $2^p = 2^p + 1$)

<u>October 25-26:</u> Suppose:

$$\mathbf{n \cdot (n+1) \cdot a_{n+1} = n \cdot (n-1) \cdot a_n - (n-2) \cdot a_{n-1} \quad (*)}$$

for every positive integer n ≥ 1. If $a_0 = 1$ and $a_1 = 2$, find:

$$\sum_{i=0}^{50} \frac{a_i}{a_{i+1}} = \frac{a_0}{a_1} + \frac{a_1}{a_2} + \cdots + \frac{a_{50}}{a_{51}}$$

<u>Solution:</u> We have:

$$n = 1 \;\Rightarrow\; 1 \cdot 2 \cdot a_2 = 1 \cdot 0 \cdot a_1 - (1-2) \cdot a_0 = a_0 = 1 \;\Rightarrow\; a_2 = \frac{1}{2}$$

$$n = 2 \;\Rightarrow\; 3 \cdot 2 \cdot a_3 = 2a_2 \;\Rightarrow\; a_3 = \frac{1}{3}a_2 = \frac{1}{3 \cdot 2} = \frac{1}{3!}$$

$$n = 3 \;\Rightarrow\; 4 \cdot 3 \cdot a_4 = 3 \cdot 2 \cdot a_3 - a_2 \;\Rightarrow\; a_4 = \frac{1}{4 \cdot 3}\left(3 \cdot 2 \cdot \frac{1}{3!} - \frac{1}{2}\right) = \frac{1}{4 \cdot 3}\left(3 \cdot 2 \cdot \frac{1}{3!} - \frac{3}{3!}\right)$$
$$= \frac{1}{4 \cdot 3}\left(\frac{3 \cdot 2 - 3}{3!}\right) = \frac{1}{4 \cdot 3} \cdot \frac{3}{3!} = \frac{1}{4!}$$

$$n = 4 \;\Rightarrow\; 5 \cdot 4 \cdot a_5 = 4 \cdot 3 \cdot a_4 - 2 \cdot a_3 \;\Rightarrow\; a_5 = \frac{1}{5 \cdot 4}\left(4 \cdot 3 \cdot \frac{1}{4!} - 2\frac{1}{3!}\right) = \frac{1}{5 \cdot 4}\left(4 \cdot 3 \cdot \frac{1}{4!} - \frac{2 \cdot 4}{4!}\right)$$
$$= \frac{1}{5 \cdot 4}\left(\frac{4 \cdot 3 - 2 \cdot 4}{4!}\right) = \frac{1}{5 \cdot 4} \cdot \frac{4}{4!} = \frac{1}{5!}$$

And it comes out as induction hypothesis

$$a_n = \frac{1}{n!}$$

Let's see if the recurring equation is fulfilled (*)

$$n \cdot (n+1) \cdot a_{n+1} = n \cdot (n+1) \cdot \frac{1}{(n+1)!} = \frac{1}{(n-1)!}$$

$$n \cdot (n-1) \cdot a_n - (n-2) \cdot a_{n-1} = n \cdot (n-1) \cdot \frac{1}{n!} - (n-2) \cdot \frac{1}{(n-1)!} = \frac{1}{(n-2)!} - \frac{n-2}{(n-1)!}$$
$$= \frac{n-1}{(n-1)!} - \frac{n-2}{(n-1)!} = \frac{1}{(n-1)!}$$

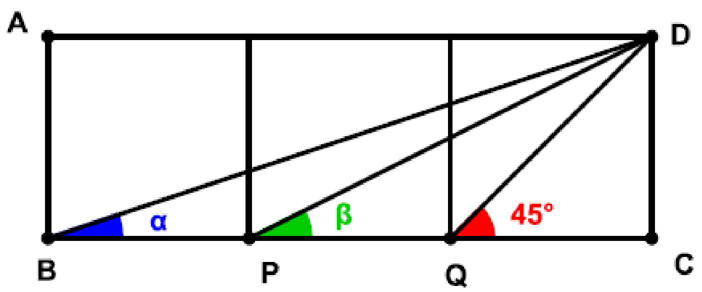
With this, we will have:

$$\frac{a_0}{a_1}+\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_{50}}{a_{51}}=\frac{1}{2}+\frac{2}{\frac{1}{2!}}+\frac{\frac{1}{2!}}{\frac{1}{3!}}+\cdots+\frac{\frac{1}{50!}}{\frac{1}{51!}}=\frac{1}{2}+\frac{4}{1}+\frac{3!}{2!}+\cdots+\frac{51!}{50!}=\frac{1}{2}+4+3+4+\cdots+51$$
$$=\frac{9}{2}+(3+4+5+\cdots+51)=\frac{9}{2}+\frac{3+51}{2}\cdot 49=\frac{2655}{2}$$

October 28: Let ABCD be a rectangle with BC = 3·AB. Prove that if P and Q are points of BC with BP = PQ = QC, then:

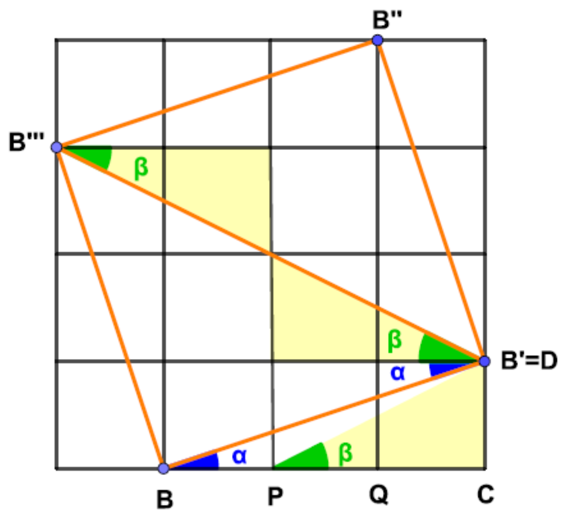
∠DBC + ∠DPC = ∠DQC

Solution: We will have, drawing the situation described in the statement:



We have to prove that $\alpha + \beta = 45°$ (since ∠DQC = 45°).

From rectangle ABCD we build a 4x4 square. From it, we build the square BB'B''B'''



The blue angles are equal by internal alternating. The angles in green are equal as the yellow triangles are equal.

We then have to $\alpha + \beta = 45°$ since B'''B' is the diagonal of square BB'B''B'''

October 29-30: Two seventh graders were allowed to play in a chess tournament for eighth graders. Each pair of participants played each other once and each participant received one point for winning each game, half for drawing, and zero points for losing. The two seventh graders received a total of eight points, and the eighth graders all received the same number of points. How many eighth grade students (at least) participated in the tournament? Is it the only solution?

Solution: If there are n eighth players, there are a total of n + 2 players in the tournament: Since each pair of players plays one game, there will have been C_2^{n+2} chess games. Since in each game a point is distributed between the two players, we will have that throughout the tournament they have been distributed

$$\binom{n+2}{2} = \frac{(n+2)\cdot(n+1)}{2}$$

points. Since the two seventh graders have obtained a total of 8 points, if each eighth grader obtains x points we will have the equation:

$$nx + 8 = \frac{(n+2)\cdot(n+1)}{2} \Rightarrow n^2 + (3-2x)n - 14 = 0 \Rightarrow n = \frac{2x - 3 \pm \sqrt{(3-2x)^2 + 56}}{2}$$

And since the equation must have at least one positive integer solution, it must be:

$$(3-2x)^2 + 56$$

a perfect square (greater than $\sqrt{56} = 7{,}48\ldots\ldots$). Let's analyse the possible cases:

$$(3-2x)^2 + 56 = 8^2 \Rightarrow 3 - 2x = \pm\sqrt{8} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 9^2 \Rightarrow 3 - 2x = \pm 5 \Rightarrow \begin{cases} x = -1 \ \text{No} \\ \mathbf{x = 4} \Rightarrow n = \frac{5 \pm 9}{2} \Rightarrow \begin{cases} \mathbf{n = 7} \\ n = -2 \ \text{No} \end{cases} \end{cases}$$

Then we already have a solution (the one with the lowest value of n): There are (n =) 7 eighth grade players and each of them has achieved 4 points in the tournament

$$(3-2x)^2 + 56 = 10^2 \Rightarrow 3 - 2x = \pm\sqrt{44} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 11^2 \Rightarrow 3 - 2x = \pm\sqrt{65} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 12^2 \Rightarrow 3 - 2x = \pm\sqrt{88} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 13^2 \Rightarrow 3 - 2x = \pm\sqrt{113} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 14^2 \Rightarrow 3 - 2x = \pm\sqrt{140} \Rightarrow x \notin \mathbb{N}$$

$$(3-2x)^2 + 56 = 15^2 \Rightarrow 3 - 2x = \pm 13 \Rightarrow \begin{cases} x = -5 \ \text{No} \\ \mathbf{x = 8} \Rightarrow n = \frac{5 \pm 9}{2} \Rightarrow \begin{cases} \mathbf{n = 14} \\ n = -1 \ \text{No} \end{cases} \end{cases}$$

Then another solution appears: there are (n =) 14 eighth grade players and each of them has got 8 points in the tournament