SOLUTIONS DECEMBER 2021

PROBLEMS FOR THIRD AND FOURTH OF E.S.O. AUTHORS: Collective "CONCURSO DE PRIMAVERA" http://www.concursoprimavera.es/#concurso.

December 1: The sum of three digits is 15. If one of them is replaced by 3, the product of the new digits is 36. What digits were there at the beginning?

Solution: Let's α, β and η the initial digits being α the one replaced by 3. Then:

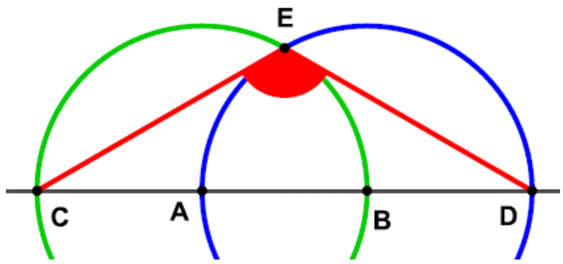
$$\alpha+\beta+\eta=15 \text{ y } 3\cdot\beta\cdot\eta=36 \;\Rightarrow\; \beta\cdot\eta=12\begin{cases}=3\cdot 4 & \Rightarrow\; \beta=3;\;\eta=4\\ =6\cdot 2 & \Rightarrow\; \beta=2;\;\eta=6\end{cases}$$

If β = 3 and η = 4, then α = 15 – (3 + 4) = 8.

If β = 2 and η = 6, then α = 15 – (2 + 6) = 7.

Then the initial digits were 8, 3, 4 or 7, 2, 6, the 8 or 7 having been changed to 3.

December 2-3: In the image there are two equal circles with centers A and B. Each of them passes through the center of the other and the line that passes through A and B cuts the circles at C and D. If E is the intersection of the two circumferences, find $\angle$CED



Solution: The triangle ΔABE is equilateral (because its sides are equal to the radius of the circumferences): Then $\angle$BAE = $\angle$ABE = 60°. Now, by the relation between central angle and inscribed angle, we have:

$$\angle \text{DCE} = \angle \text{CDE} = 30°$$

$$\text{Finally, in } \Delta\text{CDE:}$$

$$\angle \text{CED} = 180° - (\angle \text{DCE} + \angle \text{CDE}) = 180° - (30° + 30°) = 120°$$

December 4: On a math test, if each boy had scored three points higher than he did, the class average would have been 1.2 points higher than it was. Find the percentage of girls in the class.

Solution: Let x (y) be the percentage of boys (girls) and S_x (S_y) the sum of the scores of boys (girls). Then:

$$\frac{S_x+3x+S_y}{x+y}=1{,}2+\frac{S_x+S_y}{x+y}\;\Rightarrow\;3x=1{,}2(x+y)\;\Rightarrow\;1{,}8x=1{,}2y\;\Rightarrow 3x=2y$$

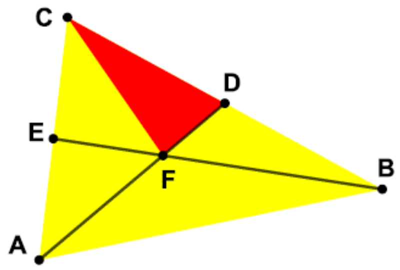
$$\Rightarrow\begin{cases}x=2k\\ y=3p\end{cases}\Rightarrow\;3\cdot 2k=2\cdot 3p\;\Rightarrow\;k=p\;\Rightarrow\begin{cases}x=2k\\ y=3k\end{cases}$$

Finally:

$$x+y=100\;\Rightarrow 2k+3k=100\;\Rightarrow k=20\;\Rightarrow\begin{cases}x=40\%\\ y=60\%\end{cases}$$

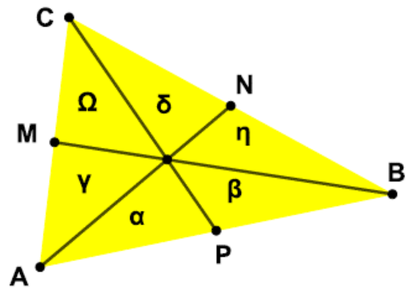
<u>December 6-7:</u> Given the triangle ΔABC, let BE and AD be two medians whose intersection is F.

Let's suppose $A_{\Delta FDC} = 3$. Find the area of triangles ΔEAB and ΔAFB and the area of quadrilateral EFDC

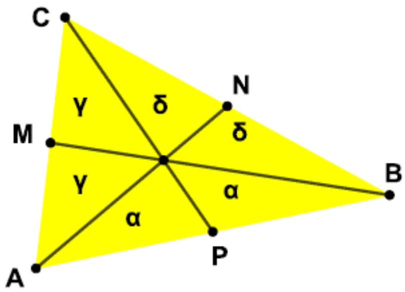


<u>Motto:</u> the medians of any triangle divide the triangle into six triangles all with the same area.

<u>Demonstration:</u> We apply that two triangles with the same base and the third common vertex (and therefore the same height) have the same area. With this we have that in the attached figure it is fulfilled:

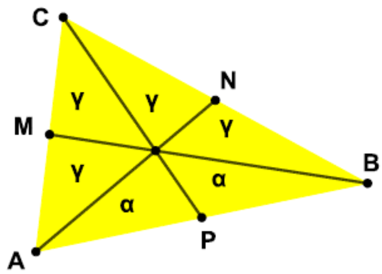


$$\alpha = \beta; \;\; \eta = \delta; \;\; \Omega = \gamma$$



For the same reason, we will have that the triangles ΔAPC and ΔPBC have the same area, so:

$$2\gamma + \alpha = 2\delta + \alpha \;\; \Rightarrow \;\; \gamma = \delta$$



Finally, in the illustration to the right, triangles ΔAMB and ΔCMB have equal area. So:

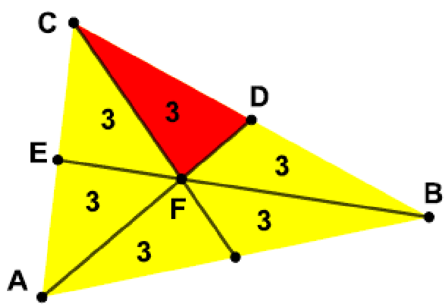
$$\gamma + 2\alpha = 3\gamma \;\; \Rightarrow \;\; \gamma = \alpha$$

And so all six triangles have the same area.

<u>Solution:</u> Applying the previous lemma to the triangle of the statement, we will have the attached figure.

From it we obviously get:

$$A_{\Delta EAB} = 3 + 3 + 3 = 9$$
$$A_{\Delta AFB} = 3 + 3 = 6$$
$$A_{EFDC} = 3 + 3 = 6$$



<u>December 8:</u> Find the natural numbers less than 100 with the greatest number of divisors.

<u>Solution:</u> Recall that if the natural N is given with factorial decomposition as a product of primes:

$$N = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \cdots \cdot p_n^{\alpha_n}$$

then the divisors of N are the numbers of the form:

$$d = p_1^{\beta_1} \cdot p_2^{\beta_2} \cdot \cdots \cdot p_n^{\beta_n}$$

where:

$$0 \leq \beta_i \leq \alpha_i \quad \forall i \in \{1, 2, \cdots, n\}$$

so the number of divisors of N is:

$$(\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdot \cdots \cdot (\alpha_n + 1)$$

Since we are looking for numbers less than 100, these can have up to six twos, no more than four threes, at most two fives or two sevens. For size it is better to have more twos than threes or as many threes as fives and sevens. As 2·3·5·7 exceeds 100 we will start from doses and see how we accompany them:

$2^6 = 64$, have (6 + 1 =) 7 divisors.

$2^5 \cdot 3 = 96$, have ((5 + 1)·(1 + 1) =) 12 divisors.

$2^4 \cdot 3 = 48$ y $2^4 \cdot 5 = 80$, have ((4 + 1)·(1 + 1) =) 10 divisors.

$2^3 \cdot 3^2 = 72$, $2^2 \cdot 3 \cdot 5 = 60$, $2^2 \cdot 3 \cdot 7 = 84$, $2 \cdot 3^2 \cdot 5 = 90$, have 12 divisors.

Then the answer to the statement is: the greatest number of divisors of the numbers of one or two digits is 12 and the numbers that reach them are: 96, 72, 60, 90 and 84.

December 9: I have two dice, one red and one blue. If I roll both at the same time, what is the probability that the number on the red die is greater than the number on the blue die?

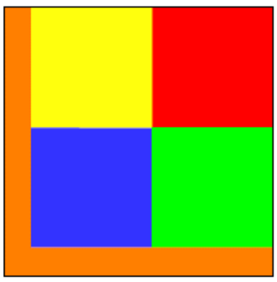
Solution:

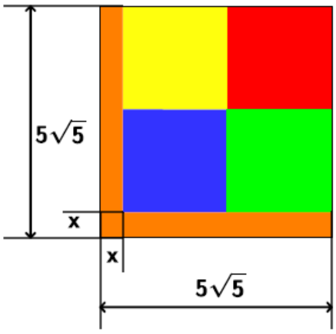
A\R	1	2	3	4	5	6
1		*	*	*	*	*
2			*	*	*	*
3				*	*	*
4					*	*
5						*
6						

In the attached table the different possible results of the two launches are collected and with a star the results favourable to the event of the statement. we will have then:

$$P = \frac{\text{number of favorable cases}}{\text{number of possible cases}} = \frac{5+4+3+2+1}{36} = \frac{5}{12}$$

December 10-11: We divide a square with an area of 125 cm² into five regions, four squares and one L-shaped polygon, all of equal area. What is the length, in cm, of the shortest side of the L-shaped polygon?





Solution: Since the area of the initial square is 125 cm^2, its side is:

$$\sqrt{125} = 5\sqrt{5}\ \mathrm{cm}^2$$

Let x be the smallest side of the L-shaped polygon with area 25 cm^2; then:

$$5\sqrt{5}x + x(5\sqrt{5} - x) = 25 \Rightarrow x^2 - 10\sqrt{5}x + 25 = 0 \Rightarrow \begin{cases} x = 5\sqrt{5} - 10 \\ x = 5\sqrt{5} + 10 \end{cases}$$

The second solution is neglected (because it is greater than the side of the square), so $x = 5\sqrt{5} - 10$

December 13: Find the natural numbers whose square and the number itself end in the same two digits and in the same order.

Solution: Suppose N and N^2 end in the digits a and b (and in this order). Then:

$$N = 100x + 10a + b$$

$$N^2 = (100x + 10a + b)^2 = 100y + 20ab + b^2$$

(where x (y) do not have to be digits, but represent numbers, more exactly x are the digits of N except the units and tens and similarly y). Namely:

b^2 ends in the same number that is b

$20ab + b^2$ ends in $10a + b$

The first condition leads to b = 0 (b^2 = 0), b = 1 (b^2 = 1), b = 5 (b^2 = 25) y b = 6 (b^2 = 36).

If b = 0

$$\begin{matrix} 20ab + b^2 \text{ ends in } 00 \\ 10a + b \text{ ends in } 10a \end{matrix} \Rightarrow a = b = 0 \Rightarrow \text{N ends in } 00$$

If b = 1

$$\begin{matrix} 20ab + b^2 = 20a + 1 \\ 10a + b = 10a + 1 \end{matrix} \Rightarrow 2a \text{ y } a \text{ they end the same} \Rightarrow a = 0;\ b = 1 \Rightarrow \text{N ends in } 01$$

If b = 5

$$\begin{matrix} 20ab + b^2 = 100a + 25 \\ 10a + b = 10a + 5 \end{matrix} \Rightarrow a = 2 \Rightarrow a = 2;\ b = 5 \Rightarrow \text{N ends in } 25$$

If b = 6

$$\begin{matrix} 20ab + b^2 = 120a + 36 \\ 10a + b = 10a + 6 \end{matrix} \text{ end in the same two digits}$$

$$\text{Si } a = 0 \Rightarrow \begin{cases} 120a + 36 \text{ ends in} 36 \\ 10a + 6 \text{ ends in } 06 \end{cases} \text{NO}$$

$$\text{Si } a = 1 \Rightarrow \begin{cases} 120a + 36 \text{ ends in } 56 \\ 10a + 6 \text{ ends in } 16 \end{cases} \text{NO}$$

$$\text{Si } a = 2 \Rightarrow \begin{cases} 120a + 36 \text{ ends in} 76 \\ 10a + 6 \text{ ends in } 26 \end{cases} \text{NO}$$

$$\text{Si } a = 3 \Rightarrow \begin{cases} 120a + 36 \text{ ends in } 96 \\ 10a + 6 \text{ ends in } 36 \end{cases} \text{NO}$$

$$\text{Si } a = 4 \Rightarrow \begin{cases} 120a + 36 \text{ ends in } 16 \\ 10a + 6 \text{ ends in} 46 \end{cases} \text{NO}$$

$$\text{Si a} = 5 \ \Rightarrow \begin{cases} 120\text{a} + 36 \text{ ends in} 36 \\ 10\text{a} + 6 \text{ ends in } 56 \end{cases} \ \text{NO}$$

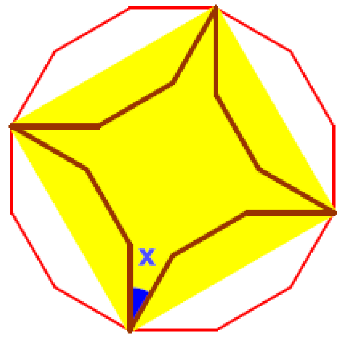
$$\text{Si a} = 6 \ \Rightarrow \begin{cases} 120\text{a} + 36 \text{ ends in } 56 \\ 10\text{a} + 6 \text{ ends in } 66 \end{cases} \ \text{NO}$$

$$\text{Si a} = 7 \ \Rightarrow \begin{cases} 120\text{a} + 36 \text{ ends in } 76 \\ 10\text{a} + 6 \text{ ends in } 76 \end{cases} \ \text{YEAH}$$

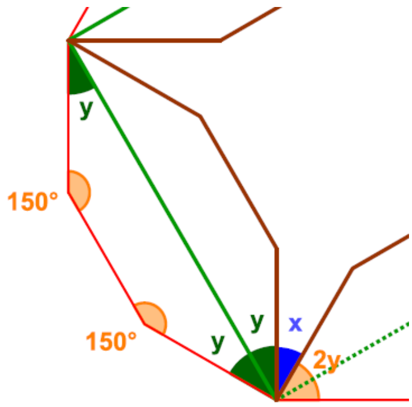
$$\text{Si a} = 8 \ \Rightarrow \begin{cases} 120\text{a} + 36 \text{ ends in } 96 \\ 10\text{a} + 6 \text{ ends in } 86 \end{cases} \ \text{NO}$$

$$\text{Si a} = 9 \ \Rightarrow \begin{cases} 120\text{a} + 36 \text{ ends in } 16 \\ 10\text{a} + 6 \text{ ends in } 96 \end{cases} \ \text{NO}$$

Then the numbers that end in 00, in 01, in 25 and in 76 fulfill the statement.



December 14-15: In a regular dodecagon we have inscribed a square, as shown in the figure. Furthermore, we have drawn the symmetry of the sides of the dodecagon with axis of symmetry the sides of the square. Find the measure of the angle x and the area of the star if the side of the dodecagon measures 1

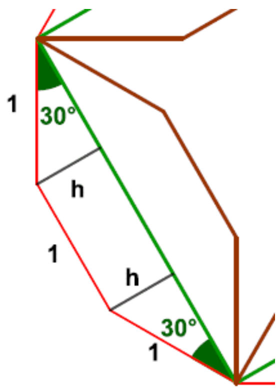


Solution: The angle between two consecutive edges of the dodecagon is equal to

$$180° - \frac{360°}{12} = 150° \ \Rightarrow \ y = \frac{360° - 2 \cdot 150°}{2} = 30°$$

Finally:

$$150° = 4y + x = 120° + x \ \Rightarrow \ x = 30°$$



To calculate the area of the star we will subtract from the area of the square the sum of the areas of the four isosceles trapezoids that generate the points of the star. We will have:

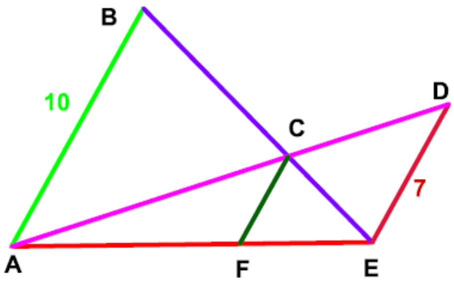
$$h = \text{sen } 30^{\circ\circ} = \frac{1}{2}$$

$$A = \frac{1 + (1 + 2\cos 30°)}{2} \cdot h = \frac{1 + 1 + \sqrt{3}}{2} \cdot \frac{1}{2} = \frac{2 + \sqrt{3}}{4}$$

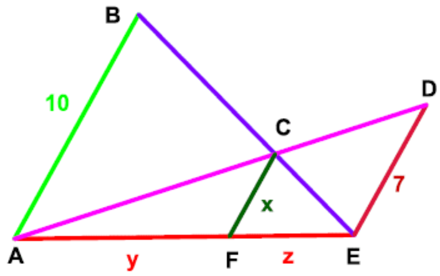
Finally:

$$A_{\text{star}} = \left(1+\sqrt{3}\right)^2 - 4\cdot\frac{2+\sqrt{3}}{4} = 2+\sqrt{3}$$

December 16-17: In the attached figure segments AB, CF and ED are parallel. If the length of AB is 10 and the length of ED is 7, find the length of segment CF.



Solution: Let x be the length of segment FC, y that of segment AF, and z that of segment FE. Then:



$$\left.\begin{array}{l} \Delta BAE \approx \Delta CFE \Rightarrow \dfrac{y+z}{10} = \dfrac{z}{x} \Rightarrow 7(y+z) = 70\dfrac{z}{x} \\ \Delta AED \approx \Delta ACF \Rightarrow \dfrac{y+z}{7} = \dfrac{y}{x} \Rightarrow 10(y+z) = 70\dfrac{y}{x} \end{array}\right\}$$

Adding the last two equations:

$$17(y+z) = 70\frac{y+z}{x} \quad \Rightarrow \quad x = \frac{70}{17}$$

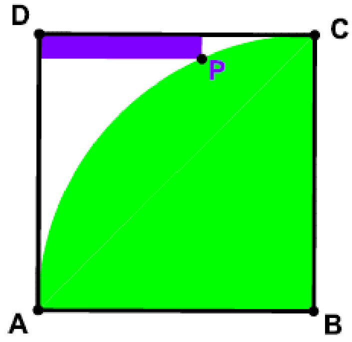
December 18: With the digits 1, 2, 3, 4 and 5 written in some order we form the number PQRST. If PQR is a multiple of 4, QRS is a multiple of 5, and RST is a multiple of 3, find the PQRST number.

Solution: If QRS must be a multiple of 5, then S = 5 (since S cannot be 0). If PQR is a multiple of 4, then QR must be a multiple of 4. Reviewing the multiplication table of 4 and remembering that the participating digits must be limited to 1, 2, 3 and 4, there are three possibilities:

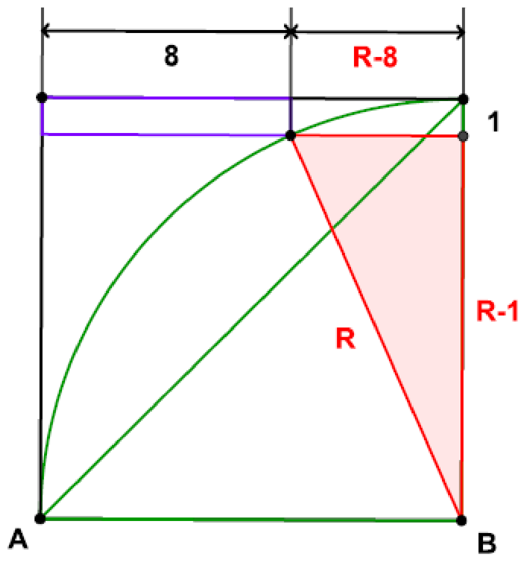
A. (4·3 =) 12 ⇒ Q =1, R = 2
B. (4·6 =) 24 ⇒ Q = 2, R = 4
C. (4·8 =) 32 ⇒ Q = 3, R = 2

Let's analyse each of these possibilities, requiring that RST must be a multiple of 3:

A. Q = 1, R = 2, S = 5 ⇒ RST = 25T must be a multiple of 3 ⇒ 7 + T must be a multiple of 3 ⇒ 7 + T must be 9 (12) and therefore T = 2 =R NO (T = 5 = S NO)
B. Q = 2, R = 4, S = 5 ⇒ RST = 45T must be a multiple of 3 ⇒ 9 + T must be a multiple of 3 ⇒ 9 + T must be 12 and therefore T = 3. The number is 12453
C. Q = 3, R = 2, S = 5 ⇒RST = 25T must be a multiple of 3 ⇒ 7 + T must be a multiple of 3 ⇒ 7 + T must be 9 (12) and therefore T = 2 =R NO (T = 5 = S NO)



December 20-21: There is a square ABCD and a quadrant with radius CB and center B. P is a point on the quadrant that is 8 units away from side DA and one unit from side DC. Find the side of the square.



Solution: Let R be the side of the initial square. We generate the red triangle in the illustration and apply Pythagoras to it. With that:

$$R^2 = (R-1)^2 + (R-8)^2 \;\Rightarrow\; R^2 - 18R + 65 = 0$$

$$\Rightarrow \begin{cases} R = 13 \\ R = 5 \end{cases}$$

Therefore the side of the square is 13 (It cannot be 5 because then a side of the triangle would be R – 8 = 5 – 8 = – 3)

December 22: Of the natural N it is known that it is a multiple of p, but it is not a multiple of 2p. Find the remainder of N when divided by 2p.

Solution: Since N is a multiple of p, we will have:

$$N = k \cdot p$$

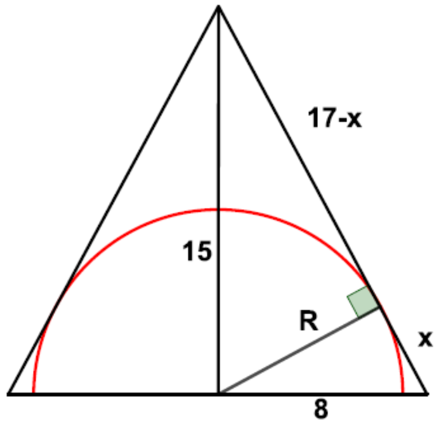
where k is odd, since if it were even: k = 2n, then:

$$N = 2 \cdot n \cdot p$$

And so, we would have that N is a multiple of 2p, which contradicts the statement. Let k = 2n + 1, then:

$$N = p \cdot (2n+1) = 2p \cdot n + p \;\Rightarrow\; r_{2p}(N) = r_{2p}(2p \cdot n) + r_{2p}(p) = 0 + p = p$$

December 23: We inscribe a semicircle in an isosceles triangle with base 16 and height 15, as shown in the figure. Find the radius of the semicircle.



Solution: The equal sides of the isosceles triangle measure:

$$\sqrt{15^2 + 8^2} = 17$$

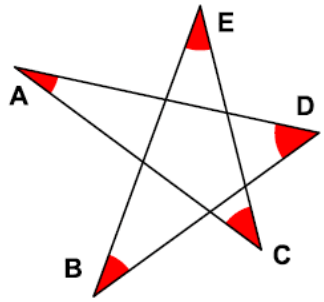
Applying Pythagoras to the two right triangles in the illustration, we will have:

$$\left.\begin{array}{l} 15^2 = R^2 + (17-x)^2 = R^2 + x^2 + 289 - 34x \\ 8^2 = R^2 + x^2 \end{array}\right\}$$

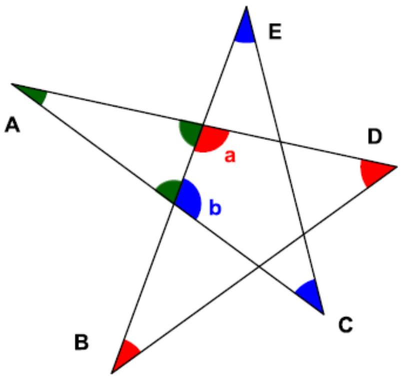
$$\Rightarrow \quad 15^2 = 8^2 + 289 - 34x \quad \Rightarrow \quad x = \frac{64}{17}$$

And substituting in the second equation:

$$R^2 = 64 - \left(\frac{64}{17}\right)^2 \Rightarrow \quad R = \frac{120}{17}$$



December 24-25: How much is the sum of the measure of the angles A, B, C, D and E of the star in the attached figure?



Solution: In the red angled triangle, we have:

$$\hat{B} + \hat{D} + a = 180° \quad \Rightarrow \quad -a = \hat{B} + \hat{D} - 180°$$

In the triangle of blue angles, we have:

$$\hat{C} + \hat{E} + b = 180° \quad \Rightarrow \quad -b = \hat{C} + \hat{E} - 180°$$

In the triangle of green angles, we have:

$$\hat{A} + (180° - a) + (180° - b) = 180°$$

And substituting –a and –b, we get:

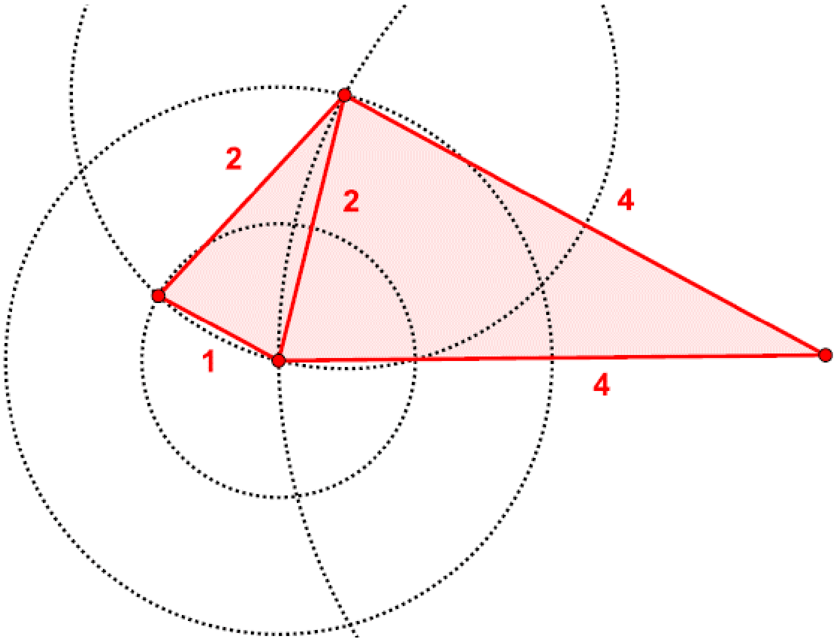
$$\hat{A} + \left(180° + \hat{B} + \hat{D} - 180°\right) + \left(180° + \hat{C} + \hat{E} - 180°\right) = 180°$$

Namely:

$$\hat{A} + \hat{B} + \hat{D} + \hat{C} + \hat{E} = 180°$$

December 27: Two sides of a quadrilateral measure 4 and 1. One of the diagonals, of length 2, divides the quadrilateral into two isosceles triangles. Calculate the perimeter of the quadrilateral.

Solution:



Since the given sides (1 and 4) and the diagonal (2) do not form a triangle (because they do not satisfy the triangle inequality: 4 > 2 + 1), the given sides and the diagonal must be concurrent. Therefore, the sides of the quadrilateral must be: 1, 4, 4 and 2. Then the perimeter is (1 + 4 + 4 + 2 =) 11.

December 28-29: Fill the cells of the adjoining matrix with digits so that all rows add up to the same amount, all columns add up to the same amount, even though the sum of a row may be different from the sum of a column

2	4		2
	3	3	
6		1	

Solution: We will have, from the attached illustration:

$$\begin{cases} 7+y=4+x \ \Rightarrow \ x=3+y \\ 8+x=7+y+z \Rightarrow 8+(3+y)=7+y+z \Rightarrow z=4 \end{cases}$$

2	4	x	2	8+x
	3	3		
6	y	1	z	7+y+z
	7+y	4+x		

2	4	x	2	
u	3	3	t	6+u+t
6	y	1	4	11+y
8+u	7+y		6+t	

Once z has been replaced by its value, we will have, in the total number of columns:

$$8+u=7+y=6+t \ \Rightarrow 1+y=t$$

and in the total rows:

$$6+u+t=11+y \Rightarrow 6+u+t=10+t \Rightarrow u=4$$

2	4	x	2	
4	3	3	t	
6	y	1	4	
12	7+y	4+x	6+t	

And since the column totals must match:

$$7 + y = 12 \;\Rightarrow\; y = 5$$
$$4 + x = 12 \;\Rightarrow\; x = 8$$
$$6 + t = 12 \;\Rightarrow\; t = 6$$

which provides the ultimate solution:

2	4	8	2	16
4	3	3	6	16
6	5	1	4	16
12	12	12	12	48

<u>December 30:</u> Calculate the remainder of dividing

$$\mathbf{x^{100} - 2x^{99} + 4}$$

by

$$\mathbf{x^2 - 3x + 2}$$

<u>Solution:</u> Let C(x) and mx + n be the quotient and the remainder of the proposed division. Then:

$$x^{100} - 2x^{99} + 4 = C(x) \cdot (x^2 - 3x + 2) + mx + n = C(x) \cdot (x - 1) \cdot (x - 2) + mx + n$$

Giving x the value 1, we have:

$$1^{100} - 2 \cdot 1^{99} + 4 = 1 - 2 + 4 = 3 = C(1) \cdot (1 - 1) \cdot (1 - 2) + m + n = 0 + m + n = m + n$$

Giving x the value 2, we will have:

$$2^{100} - 2 \cdot 2^{99} + 4 = 4 = C(2) \cdot (2 - 1) \cdot (2 - 2) + 2m + n = 0 + 2m + n = 2m + n$$

Finally:

$$\left.\begin{array}{r} 3 = m + n \\ 4 = 2m + n \end{array}\right\} \;\Rightarrow\; m = 1;\;\; n = 2$$

Namely; the remainder of the division is: x + 2

<u>December 31:</u> A bag contains 3 red balls and 2 green balls. We remove, one by one and without returning, balls from the bag until we have removed all the ones of the same colour. What is the probability that we have drawn all 3 red balls?

<u>Solution:</u> The different and mutually exclusive ways of verifying the event of the statement are:

RRR; RRVR; RVRR; VRRR

The probability of verifying each of these ways is:

$$P(RRR) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}$$

$$P(RRVR) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{10}$$

$$P(RVRR) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{10}$$

$$P(VRRR) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{10}$$

Then the required probability is:

$$\frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$