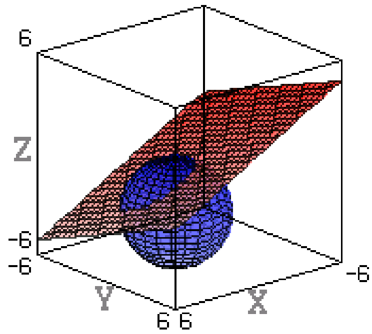
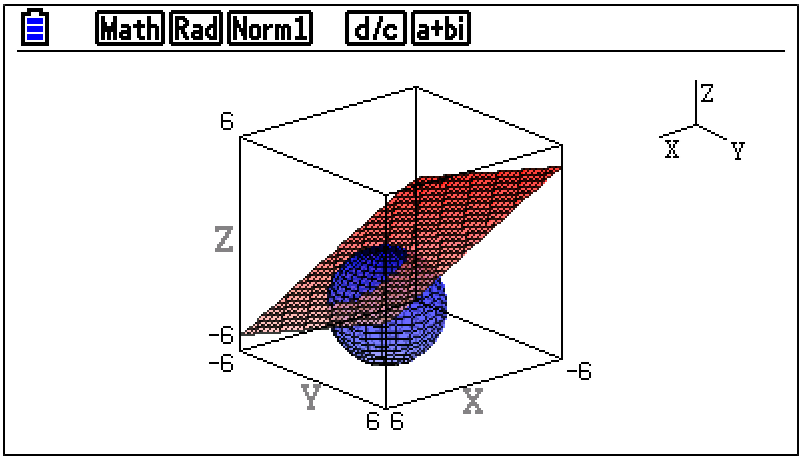
SOLUTIONS APRIL 2022

PROBLEMS WITH CALCULATOR CASIO fx-CG50. AUTHOR: RICARD PEIRÓ I ESTRUCH. IES "Abastos". València

April 1-2: Let be the sphere of equation $\mathbf{E \equiv x^2 + y^2 + z^2 - 2x + 6z = 0}$. Determine the coordinates of the center and the measure of the radius. Check if the plan $\boldsymbol{\Pi \equiv 3x - 2y + 6z + 1 = 0}$ and the sphere are secant. Determine the radius of the circle intersection of $\mathbf{E}, \boldsymbol{\Pi}$. Determine the center of the circle intersection of $\mathbf{E}, \boldsymbol{\Pi}$



Solution: We open the *Menú Gráfico 3D*. Let us represent the sphere and the plane:



Completing squares:

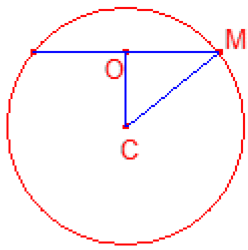
$$E \equiv (x-1)^2 + y^2 + (z+3)^2 = \left(\sqrt{10}\right)^2$$

The coordinates of the center are $C(1, 0, -3)$ and the radius $R = \sqrt{10}$.

To study the relative position of the plane and the sphere, let us calculate the distance from the center of the sphere to the plane.

$$d(C, \Pi) = \left|\frac{3 \cdot 1 - 2 \cdot 0 + 6(-3) + 1}{\sqrt{3^2 + (-2)^2 + 6^2}}\right| = 2$$

$$d(C, \Pi) = 2 < R = \sqrt{10}$$



That is, the plane and the sphere are secant.

Consider the intersection circumference of the sphere and the plane. Let O be the center of the circle. Let be the section of the sphere passing through the center C and perpendicular to the plane. Let $r = \overline{OM}$ the radius of the intersection circle.

$$\overline{CO} = 2, \overline{CM} = \sqrt{10}$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{COM}$

$$r^2 = \left(\sqrt{10}\right)^2 - 2^2 = 6 \quad r = \sqrt{6}$$

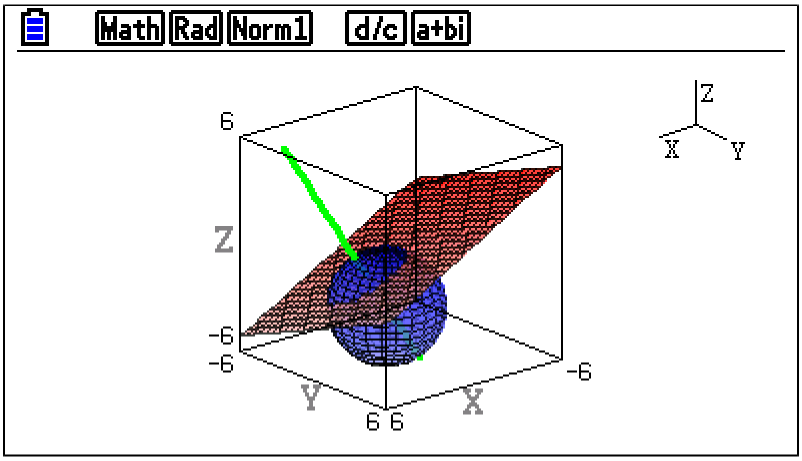
To calculate the center of the intersection circle of E,Π, let us determine the intersection of the line perpendicular to Π that passes through the center C(1,0,-3) of the sphere and the plane Π

The direction vector of the line perpendicular to the plane passing through C is the characteristic vector of the plane $v = (3, -2, 6)$.

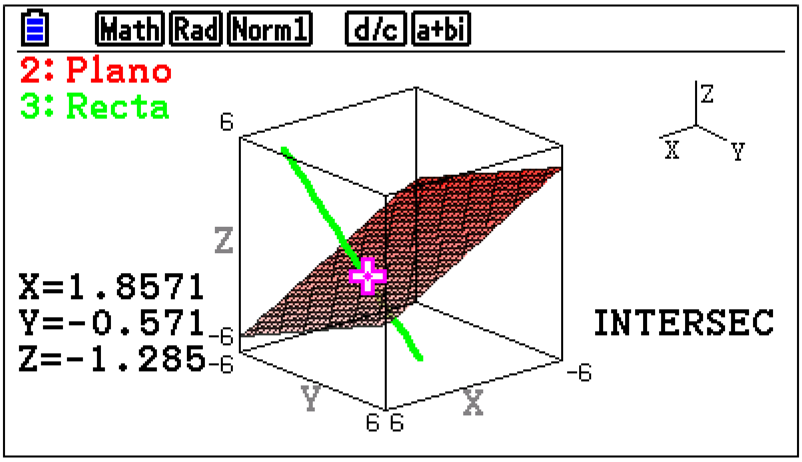
Its vector equation is:

$$r \equiv (x, y, z) = (1, 0, -3) + (3, -2, 6)\alpha$$

We open the *Menú Gráfico 3D*.
Let's draw the line r:



With the function *G-Solv* determine the intersection of the line r and the plane Π.



The center of the circle is:

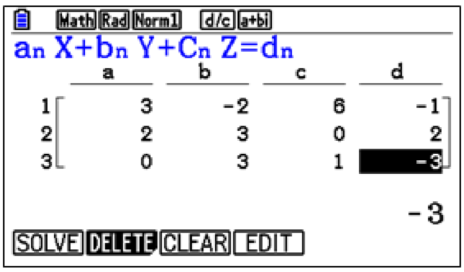
$$P(1.8571, -0.571, -1.285)$$

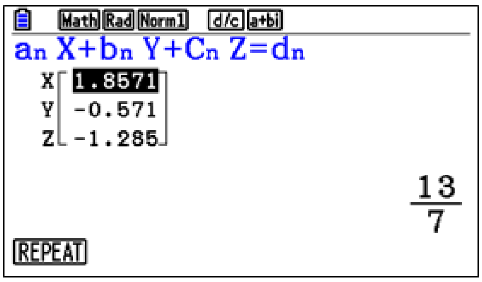
Analytically, to calculate the intersection point, we will solve the system formed by the line r and the plane Π.

$$r \equiv \begin{cases} 2x + 3y = 2 \\ 3y + z = -3 \end{cases}$$

$$\begin{cases} 3x - 2y + 6z = -1 \\ 2x + 3y = 2 \\ 3y + z = -3 \end{cases}.$$

We open the Menú Ecuación: We solve the system of linear equations formed by the line r and the plane Π.





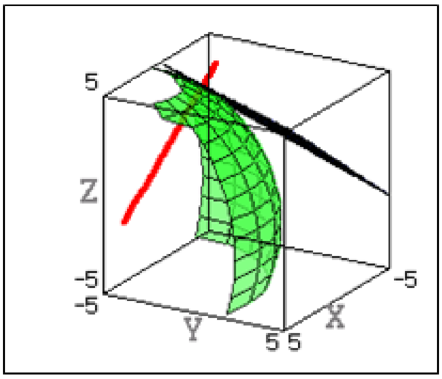
$$\begin{cases} x = \frac{13}{7} \\ y = -\frac{4}{7} \\ z = -\frac{9}{7} \end{cases}$$

So the center of the circle is:

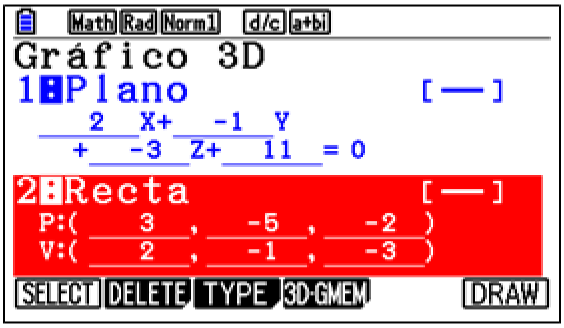
$$P\left(\frac{13}{7},-\frac{4}{7},-\frac{9}{7}\right)$$

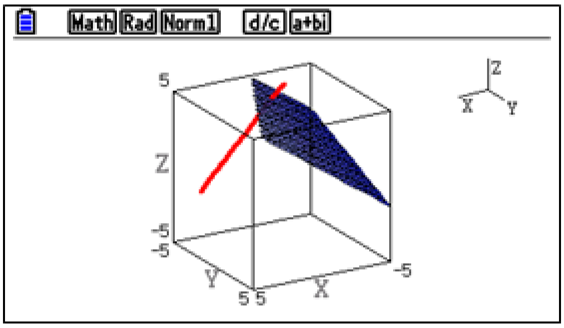
April 4: Determine the equation of the sphere with center C(3, -5, -2) and tangent to the plane

$$2x-y-3z+11=0$$

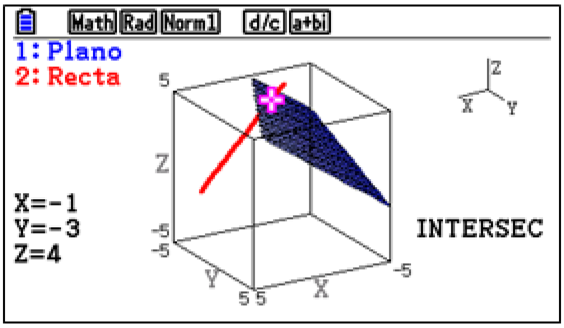


Solution: We open the *Menú Gráfico 3D*. We define the plane and the line that passes through the point C(3,-5,-2) and has the director vector the characteristic of the plane $a=(2,-1,-3)$. Line perpendicular to the plane. The intersection of the line and the plane gives us the point of tangency of the sphere and the plane.





Whit the function *G-Solv,* find the intersection of the line and the plane.



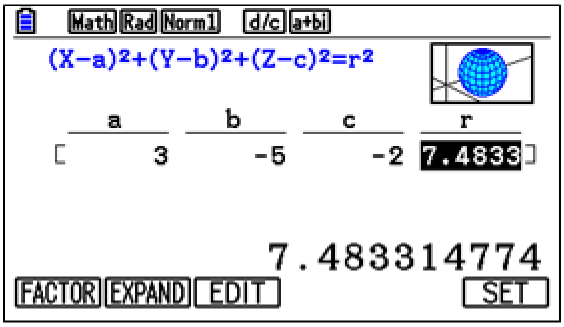
The point of tangency is T(-1,3,4). The radius of the sphere is the distance between the center and the point of tangency:

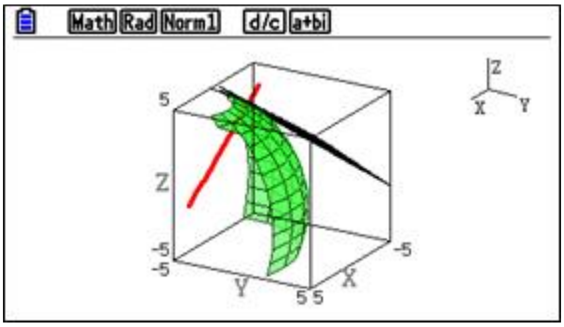
$$r=\sqrt{(-4)^2+2^2+6^2}=2\sqrt{14}$$

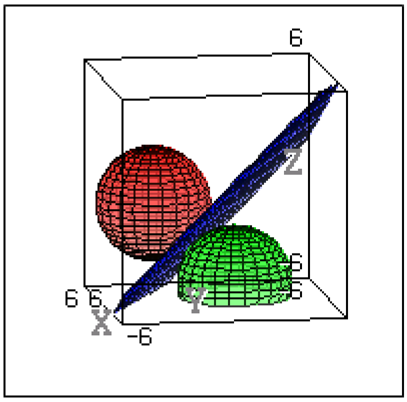
The equation of the sphere is:

$$E\equiv(x-3)^2+(y+5)^2+(z+2)^2=\left(2\sqrt{14}\right)^2$$

We define the equation of the sphere and represent it graphically.







April 5: Determine the equation of the sphere of radius r=3, which is tangent to the plane

$$x + 2y + 2z + 3 = 0$$

on the point $A(1, 1, -3)$

Solution: The center of the sphere lies on the line perpendicular to the plane at the point A(1, 1, -3). The straight line has as director vector the characteristic of the plane, a = (1, 2, 2). Its parametric equation is:

$$r \equiv \begin{cases} x = 1 + \alpha \\ y = 1 + 2\alpha \\ z = -3 + 2\alpha \end{cases}$$

Any point on the line r is:

$$P(1 + \alpha, 1 + 2\alpha, -3 + 2\alpha) \qquad \overrightarrow{AP} = (\alpha, 2\alpha, 2\alpha)$$

The radius of the sphere is:

$$r = \left\|\overrightarrow{AP}\right\| = 3 \qquad \sqrt{\alpha^2 + 4\alpha^2 + 4\alpha^2} = 3$$

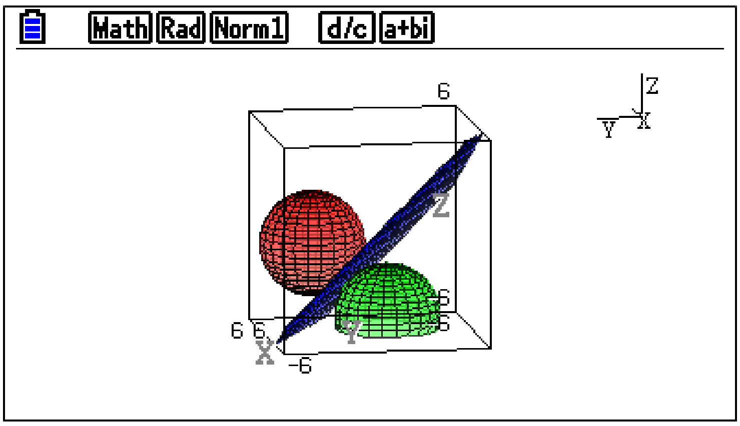
Solving the equation: $\alpha = 1, -1$. The problem has two solutions.

If $\alpha = 1$. The center sphere $O_1(2, 3, -1)$. Your equation is:

$$C_1 \equiv (x-2)^2 + (y-3)^2 + (z+1)^2 = 3^2$$

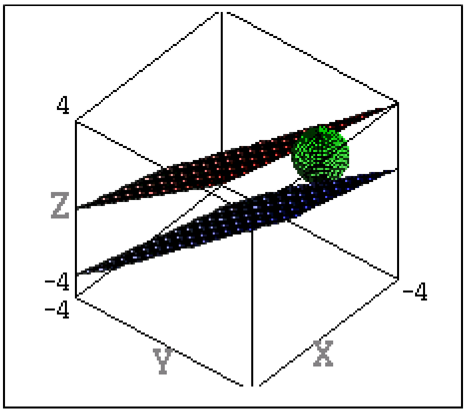
If $\alpha = -1$. The center sphere $O_1(0, -1, -5)$. Your equation is:

$$C_2 \equiv x^2 + (y+1)^2 + (z+5)^2 = 3^2$$



We open the Menú Gráfico 3D. We define the plane x + 2y + 2z + 3 = 0 and the spheres

$$C_1 \equiv (x-2)^2 + (y-3)^2 + (z+1)^2 = 3^2$$
$$C_2 \equiv x^2 + (y+1)^2 + (z+5)^2 = 3^2$$



April 6-13: A sphere has center on the line

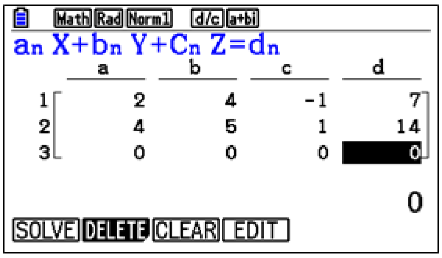
$$r \equiv \begin{cases} 2x + 4y - z - 7 = 0 \\ 4x + 5y + z - 14 = 0 \end{cases}$$

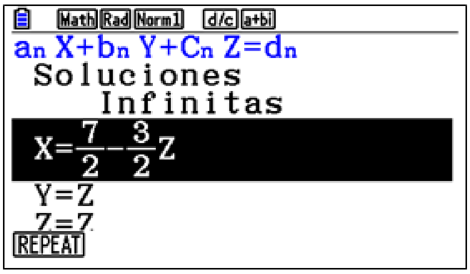
and is tangent to the planes

$$\Pi \equiv x + 2y - 2z - 2 = 0, \quad \Omega \equiv x + 2y - 2z + 4 = 0.$$

Determine your equation.

Solution: We determine the parametric equation of the line r. We open the *Menú Ecuación*



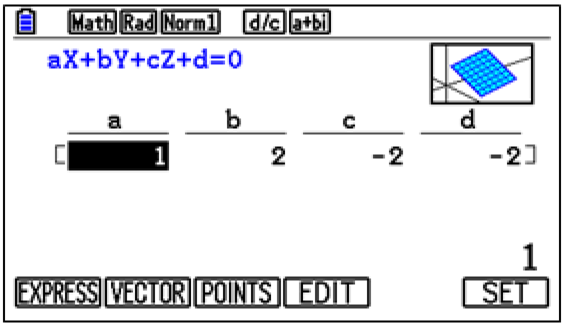


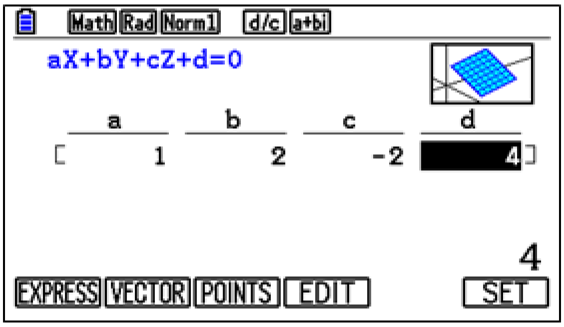
The parametric equation of the line r is:

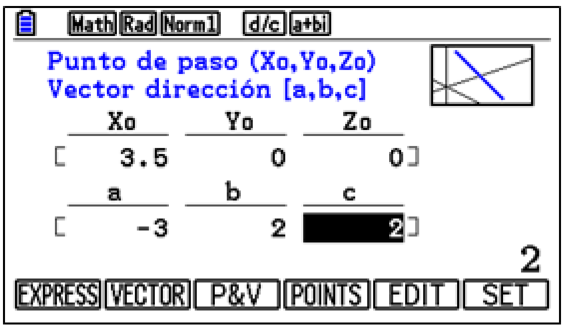
$$r \equiv \begin{cases} x = \dfrac{7}{2} - 3\alpha \\ y = 2\alpha \\ z = 2\alpha \end{cases}$$

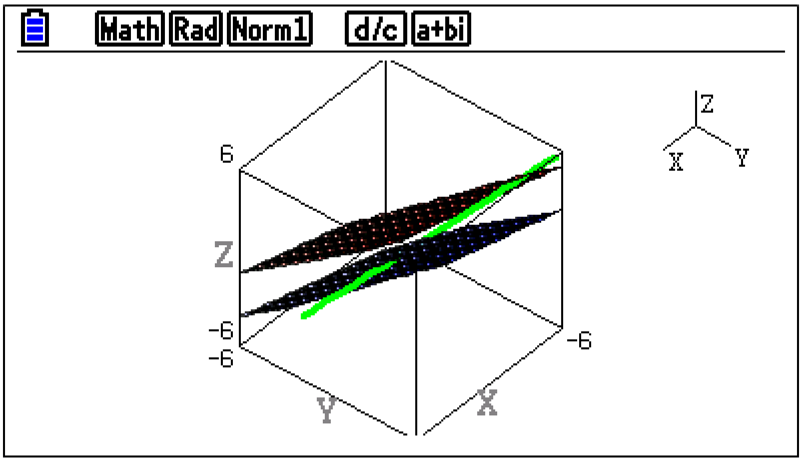
The parametric equation of the line r is $\frac{1}{1} = \frac{2}{2} = \frac{-2}{-2} \neq \frac{-2}{4}$

We open the *Menú Gráfico 3D*. We define the two planes and the line

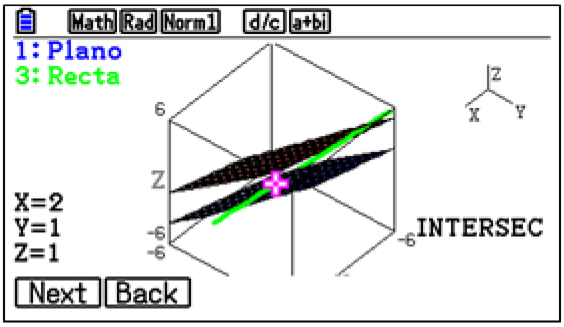


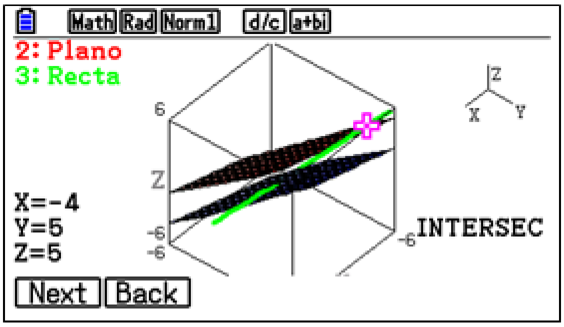






Whit the function G-*Solv,* determine the intersection of the line and each of the planes.





The coordinates of the intersection point of the plane $\Pi \equiv x + 2y - 2z - 2 = 0$ and the straight

$r \equiv \begin{cases} 2x + 4y - z - 7 = 0 \\ 4x + 5y + z - 14 = 0 \end{cases}$ is: $P(2, 1, 1)$

The coordinates of the intersection point of the plane $\Omega \equiv x + 2y - 2z + 4 = 0$ and the straight

$r \equiv \begin{cases} 2x + 4y - z - 7 = 0 \\ 4x + 5y + z - 14 = 0 \end{cases}$ is: $Q(-4, 5, 5)$

The center of the sphere is the midpoint of the segment $\overline{PQ}$. Its coordinates are: $O(-1, 3.3)$

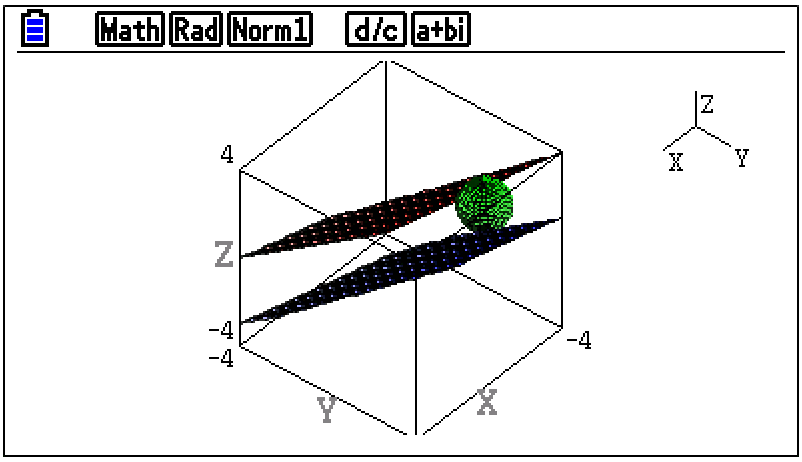
The radius is equal to the distance from the center $O(-1, 3.3)$ to the plane $\Pi \equiv x + 2y - 2z - 2 = 0$

$$r = \left| \frac{-1 + 2 \cdot 3 - 2 \cdot 3 - 2}{\sqrt{1^2 + 2^2 + (-2)^2}} \right| = 1$$

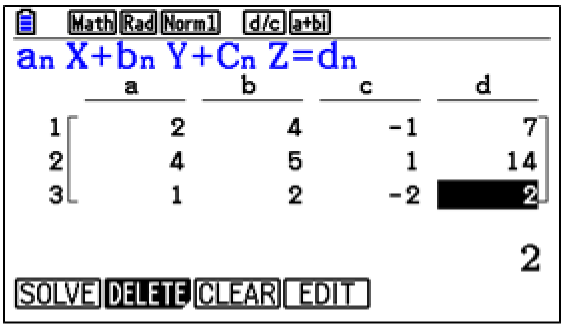
The equation of the sphere is:

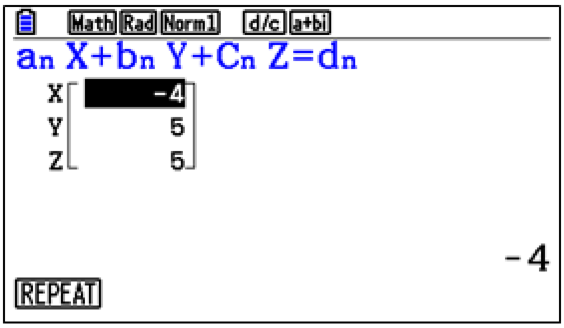
$$(x + 1)^2 + (y - 3)^2 + (z - 3)^2 = 1^2$$

We define and represent the sphere:



To calculate the intersection points of the line r and each of the planes, the systems formed by the line and each of the planes can be solved: Let's open the *Menú Ecuación*:



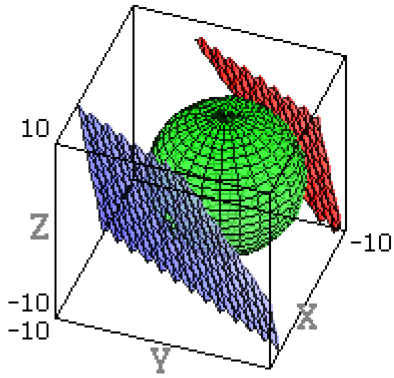


The coordinates of the intersection point of the plane $\Omega \equiv x + 2y - 2z + 4 = 0$ and the straight

$$r \equiv \begin{cases} 2x + 4y - z - 7 = 0 \\ 4x + 5y + z - 14 = 0 \end{cases} \text{is: } Q(-4, 5, 5)$$

Abril 7-8: Determine the equation of the sphere that is tangent to the planes

$\mathbf{\Pi \equiv 6x - 3y - 2z - 35 = 0}$, $\quad \mathbf{\Omega \equiv 6x - 3y - 2z + 63 = 0}$, knowing that the point M (5,-1,-1) is a point of tangency in one of the planes.



Solution: The two planes are parallel since $\frac{6}{6} = \frac{-2}{-2} = \frac{-2}{-2} \neq \frac{-35}{63}$. Note that M(5,-1,-1) belongs to the plane $\Pi \equiv 6x - 3y - 2z - 35 = 0$ since it satisfies your equation:

$$6 \cdot 5 - 3 \cdot (-1) - 2 \cdot (-1) - 35 = 0$$

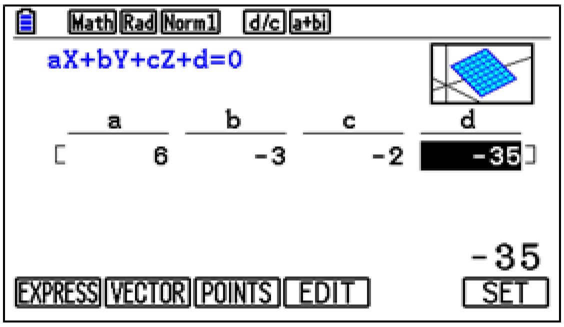
The diameter of the sphere is the distance from the point M(5,-1,-1) to the plane $\Omega \equiv 6x - 3y - 2z + 63 = 0$

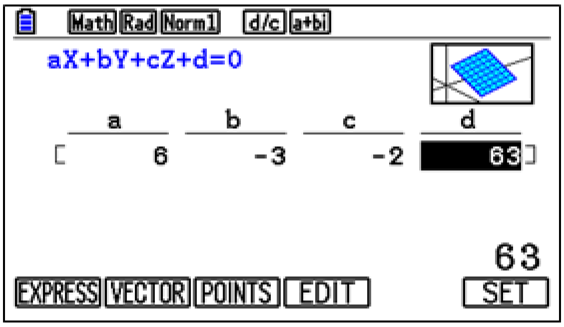
$$2r = \left| \frac{6 \cdot 5 - 3 \cdot (-1) - 2 \cdot (-1) + 63}{\sqrt{6^2 + (-3)^2 + (-2)^2}} \right| = 14$$

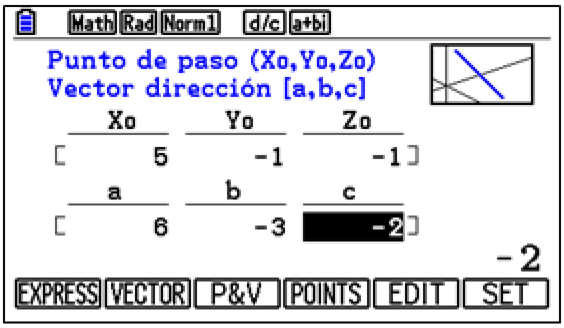
So, the radius of the sphere is r=7. The center of the sphere is the midpoint of the segment formed by the point M and the projection of M on the plane $\Omega \equiv 6x - 3y - 2z + 63 = 0$. We calculate the equation of the line perpendicular to the plane $\Omega \equiv 6x - 3y - 2z + 63 = 0$ that passes through M that has the direction vector the characteristic of the plan a=(6,-3,-2). Your equation is:

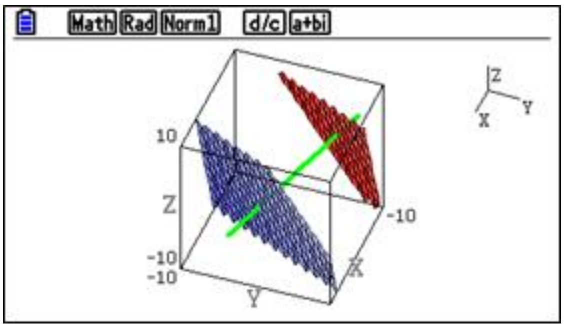
$$r \equiv (x, y, z) = (5, -1, -1) + \alpha(6, -3, -2)$$

We open the *Menú Gráfico 3D*. We define and represent the two planes and the line.

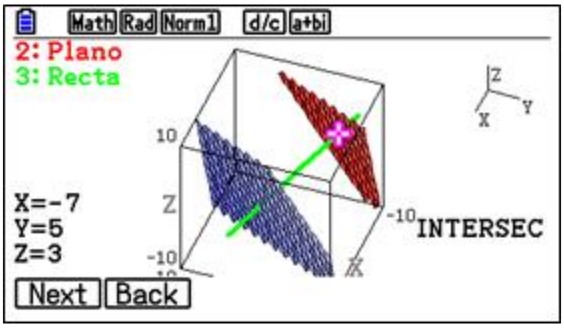








To determine the intersection of the plane $\Omega \equiv 6x - 3y - 2z + 63 = 0$ and the straight $r \equiv (x, y, z) = (5, -1, -1) + \alpha(6, -3, -2)$ function *G-Solv* is used:



The coordinates of the projection point are:

$$M'(-7, 5, 3)$$

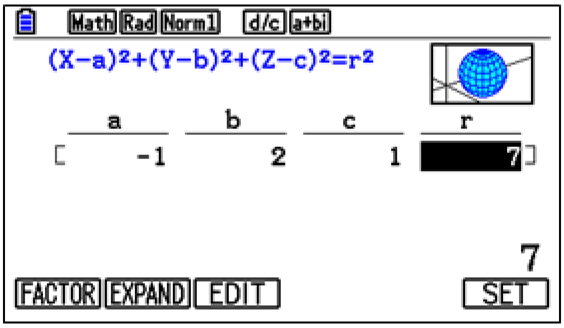
The center of the sphere is the midpoint of the segment $\overline{MM'}$. Its coordinates are:

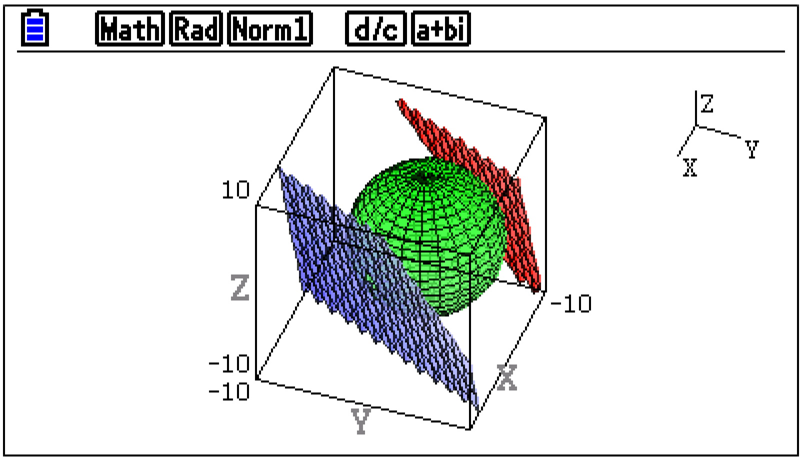
$$O(-1, 2, 1)$$

The equation of the sphere is:

$$(x + 1)^2 + (y - 2)^2 + (z - 1)^2 = 7^2$$

We open the *Menú Gráfico 3D*. We define the equation of the sphere and represent it

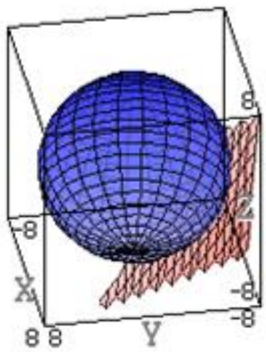




April 9: Determine the equation of the plane tangent to the sphere

$$x^2 + y^2 + z^2 = 49$$

on the point M (6, -3, -2)



Solution: The sphere has center point O(0, 0, 0) and radius r=7. The point M(6,-3,-2) belongs to the sphere since $6^2 + (-3)^2 + (-2)^2 = 49$. The characteristic vector of the tangent plane to the sphere at point M is:

$$\overrightarrow{OM} = (6, -3, -2)$$

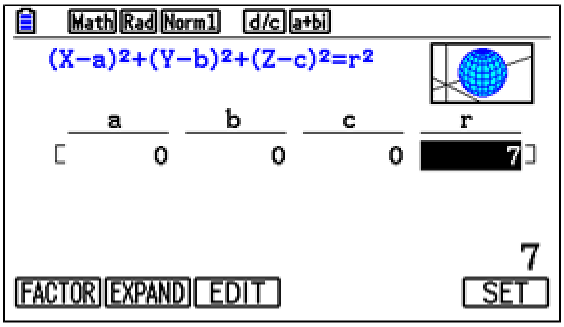
The equation of the plane is:

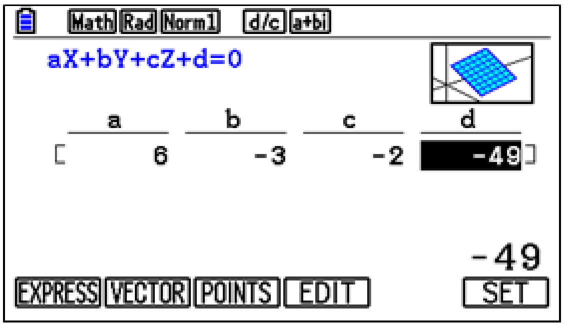
$$6(x - 6) - 3(y + 3) - 2(z + 2) = 0$$

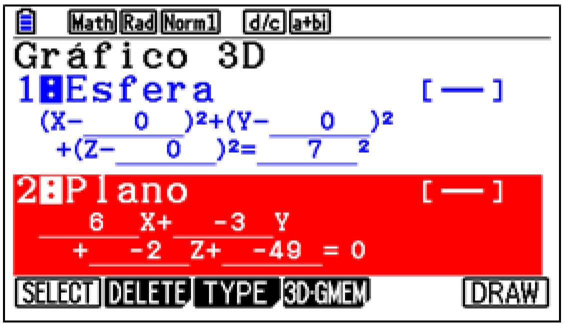
Simplifying:

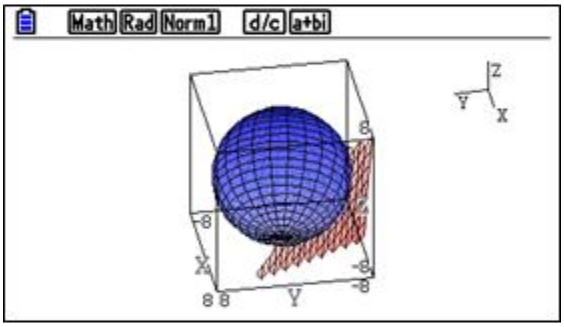
$$6x - 3y - 2z - 49 = 0$$

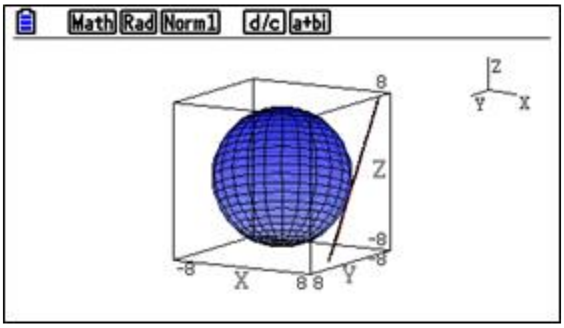
We open the *Menú Gráfico 3D*. We define the sphere and the plane and represent them:









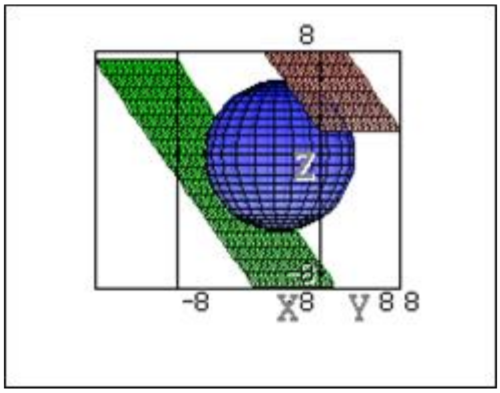


April 11: Determine the equations of the planes tangent to the sphere

$$(x-3)^2 + (y+2)^2 + (z-1)^2 = 25$$

parallel to the plane

$$4x + 3z - 17 = 0$$



Solution: The sphere has center O(3, -2, 1) and radius r = 5. The characteristic vector of the plane 4x + 3z – 17 = 0 is a = (4,0,3). The line perpendicular to the plane 4x + 3z – 17 = 0 that passes through the center of the sphere has the direction of the characteristic vector of the plane:

$$(x, y, z) = (3, -2, 1) + \alpha(4, 0, 3)$$

With the intersection of the line and the sphere we calculate the points of tangency:

$$\begin{cases} (x-3)^2 + (y+2)^2 + (z-1)^2 = 25 \\ (x, y, z) = (3, -2, 1) + \alpha(4, 0, 3) \end{cases}$$

$$(4\alpha)^2 + 0^2 + (3\alpha)^2 = 25$$

Therefore, $\alpha = 1, -1$

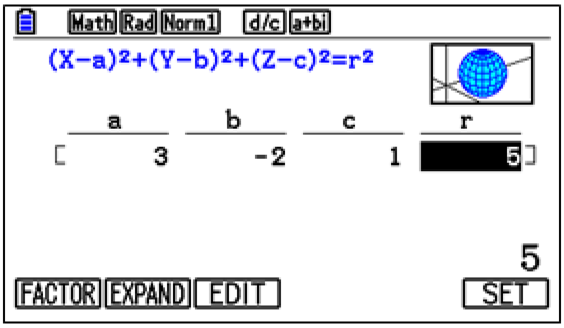
If $\alpha. = 1$, the point of tangency is $T_1(7, -2, 4)$. The equation of the plane is:

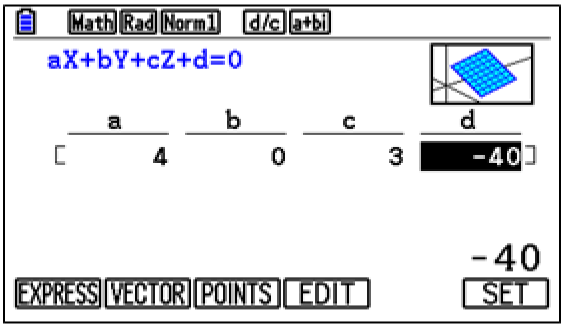
$$\pi_1 \equiv 4(x-7) + 3(z-4) = 0 \qquad \pi_1 \equiv 4x + 3z - 40 = 0$$

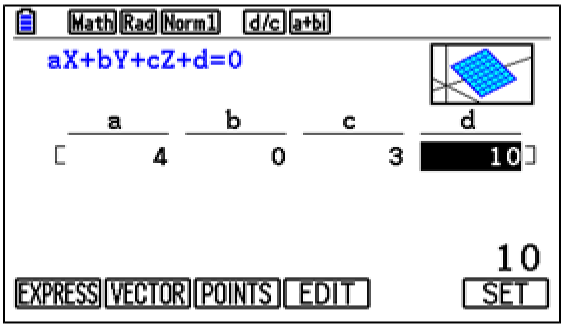
If $\alpha = -1$ the point of tangency is $T_2(-1, -2, -2)$. The equation of the plane is:

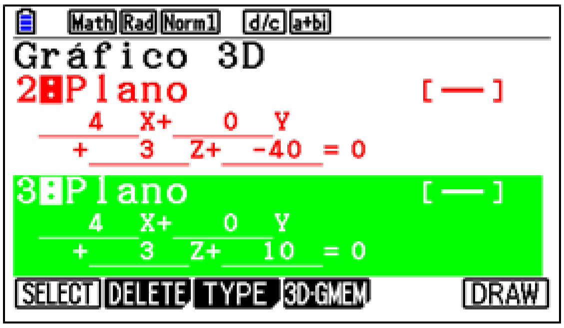
$$\pi_2 \equiv 4(x+1) + 3(z+2) = 0 \qquad \pi_2 \equiv 4x + 3z + 10 = 0$$

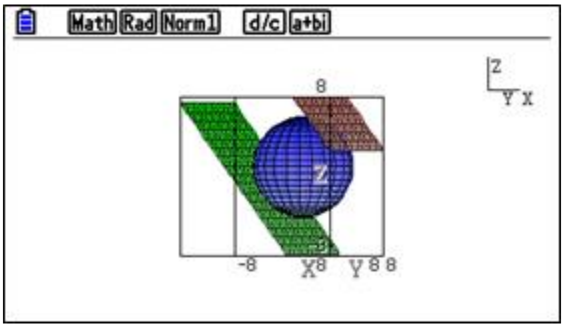
We open the *Menú Gráfico 3D*. We define the sphere and the two planes and represent them:

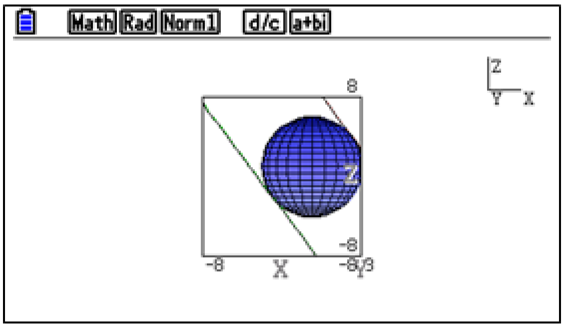








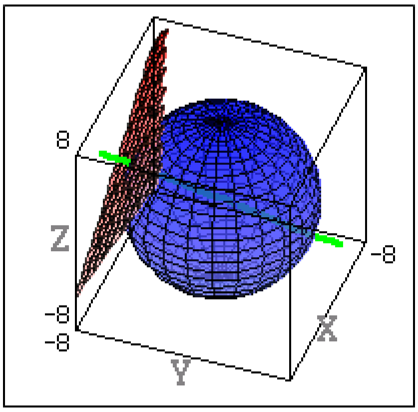




April 12: Prove that the plane 2x - 6y + 3z – 49 = 0 is tangent to the sphere

$$x^2 + y^2 + z^2 = 49$$

Calculate the coordinates of the point of tangency.



Solution: The sphere $x^2 + y^2 + z^2 = 49$ has center O(0,0,0) and radius r=7. The plane is tangent if the distance from the center of the sphere to the plane is equal to the radius. The distance from the center O(0,0,0) to the plane $2x - 6y + 3z - 49 = 0$ is:

$$d = \frac{|2 \cdot 0 - 6 \cdot 0 + 3 \cdot 0 - 49|}{\sqrt{2^2 + (-6)^2 + 3^2}} = 7$$

Therefore, the plane is tangent to the sphere. To calculate the point of tangency we will calculate the intersection of the plane 2x - 6y + 3z – 49 = 0 and the line perpendicular to the plane that passes through the center of the sphere. The line passes through the point O(0, 0, 0) and has the direction of the characteristic vector of the plane,

$$a = (2, -6, 3)$$

Your equation is

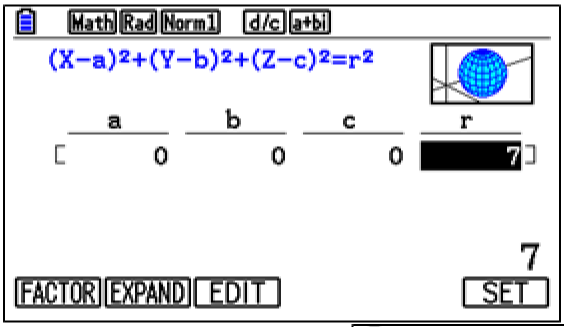
$$r \equiv (x, y, z) = (0, 0, 0) + \alpha(2, -6, 3) \qquad (x, y, z) = (2\alpha, -6\alpha, 3\alpha)$$

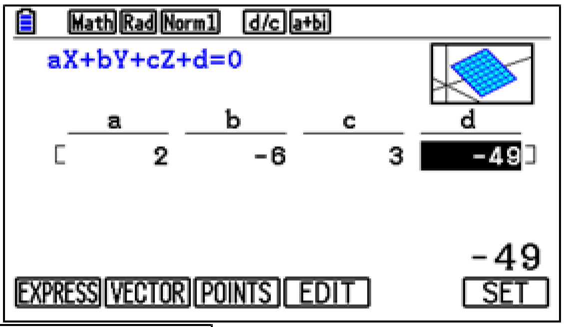
Substituting the coordinates into the equation of the plane:

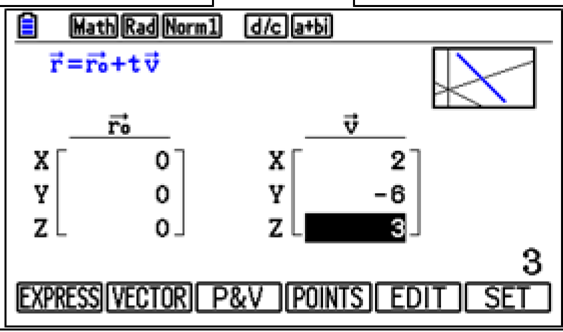
$$2 \cdot (2\alpha) - 6(-6\alpha) + 3(3\alpha) - 49 = 0$$

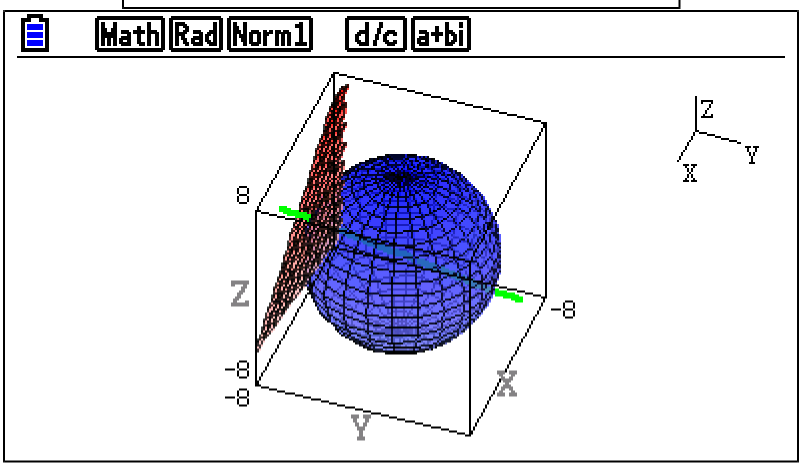
Solving the equation $\alpha = 1$. The point of tangency has coordinates $P(2, -6, 3)$

We open the *Menú Gráfico 3D*. We define and represent the sphere, the plane and the line.

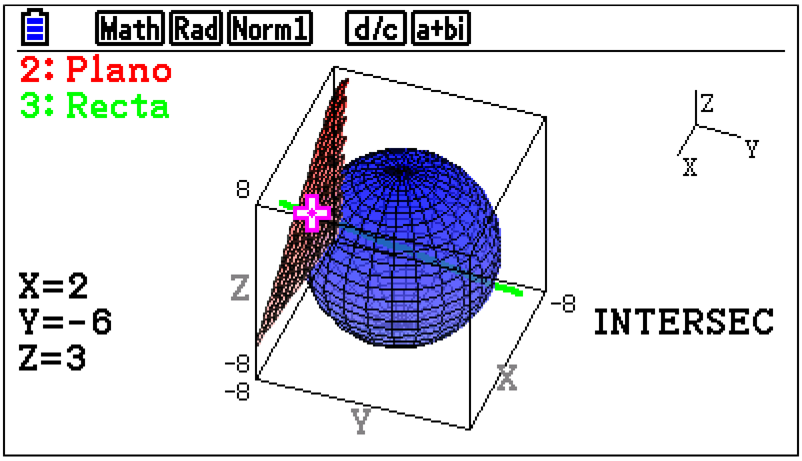






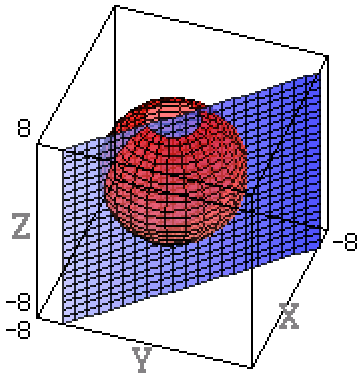


We determine with the function *G-Solv,* the intersection of the line and the plane:

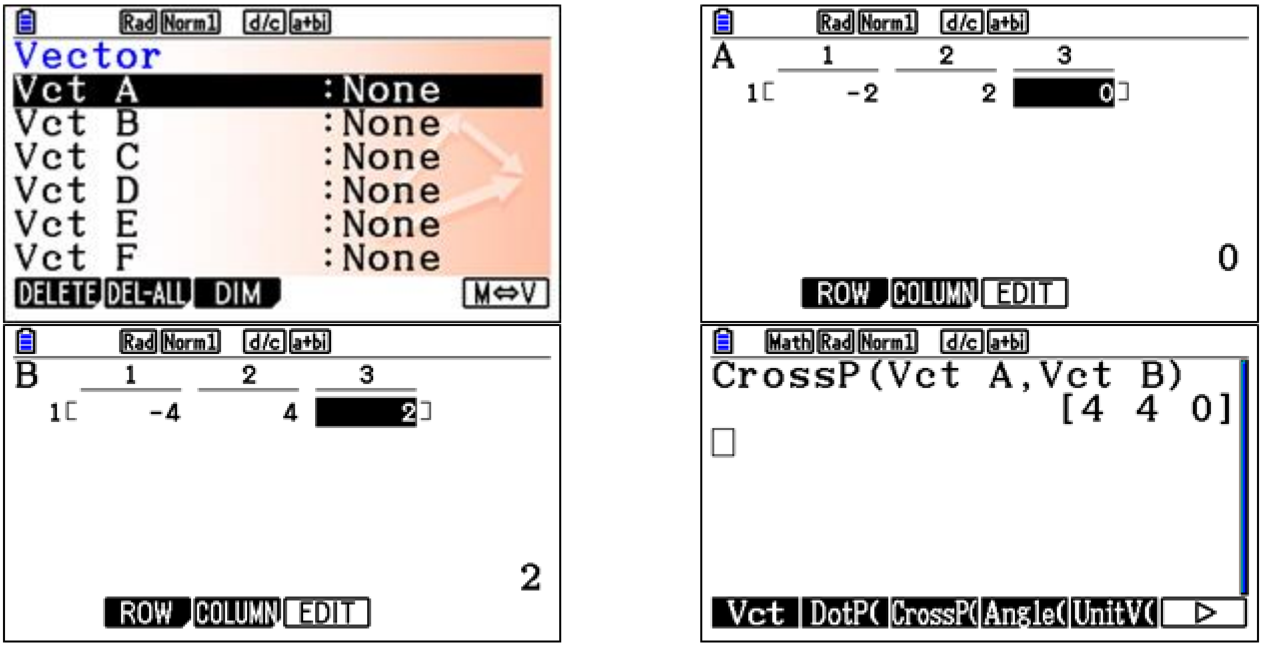


The point of tangency has coordinates $\mathrm{P}(2,-6,3)$

<u>April 14:</u> Find the equation of the circle passing through the points A (3, -1, -2), B (1, 1, -2) and C (-1, 3, 0)



<u>Solution:</u> We open the *Menú Ejec-Mat*: We define the vectors $\overrightarrow{\mathrm{AB}}=(-2,2,0), \overrightarrow{\mathrm{AC}}=(-4,4,2)$



The plane passing through A, B and C has equation:

$$\pi_{\mathrm{ABC}} \equiv \mathrm{x}+\mathrm{y}+\mathrm{D}=0$$

The point A(3, -1, -2) belongs to the plane. Then, $3-1+\mathrm{D}=0$, $\mathrm{D}=-2$

$$\pi_{\mathrm{ABC}} \equiv \mathrm{x}+\mathrm{y}-2=0$$

To determine the center of one of the spheres that pass through the points A,B,C we will determine the mediating planes of the segments $\overline{\mathrm{AB}}$, $\overline{\mathrm{AC}}$ and we will calculate the intersection of these two planes and the plane that passes through A,B,C. the middle point C_1 of the segment $\overline{AB}$ has coordinates: $C_1(2,0,-2)$. The characteristic vector of the mediating plane of the segment $\overline{AB}$ is $\overrightarrow{AB}=(-2,2,0)$. Your equation is:

$$\pi_1 \equiv -x+y+E=0$$

The point $C_1(2,0,-2)$ belongs to the plane, therefore:

$$-2+0+E=0 \quad E=2 \quad \pi_1 \equiv -x+y+2=0$$

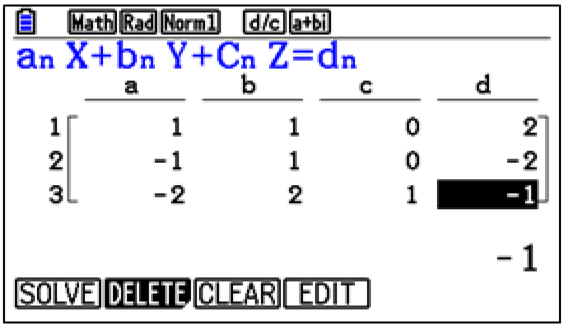
the middle Point B_1 of the segment $\overline{AC}$ has coordinates: $B_1(1,1,-1)$. The characteristic vector of the mediating plane of the segment $\overline{AC}$ is $\overrightarrow{AC}=(-4,4,2)$. Your equation is:

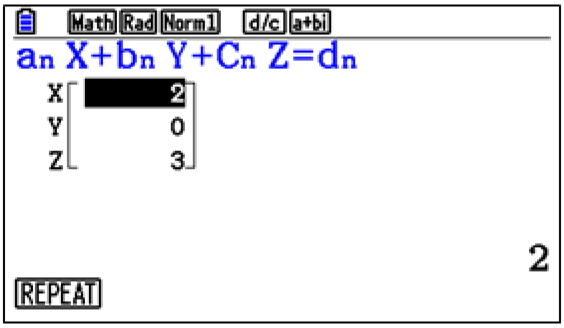
$$\pi_2 \equiv -2x+2y+z+F=0$$

The point $B_1(1,1,-1)$ belongs to the plane, therefore $-2+2-1+F=0,\ F=1,\ \pi_2 \equiv -2x+2y+z+1=0$. The center is the intersection of the three planes.

We open the *Menú Ecuación*: We solve the system:

$$\begin{cases} x+y=2 \\ -x+y=-2 \\ -2x+2y+z=-1 \end{cases}$$





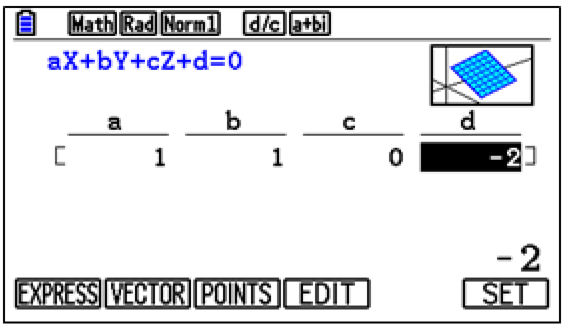
The center coordinates are: O(2, 0, 3). The radius of the circle passing through points A, B and C is:

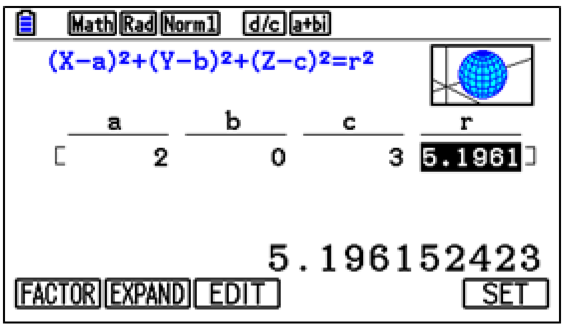
$$r=\sqrt{(3-2)^2+(-1-0)^2+(-2-3)^2}=\sqrt{27}$$

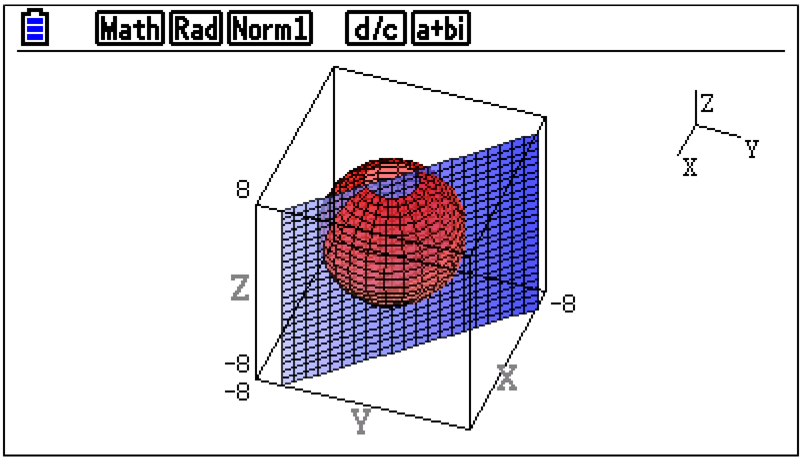
The equation of the circle is equal to the intersection of the plane $\pi_{ABC} \equiv x+y-2=0$ and the sphere, with center $O(2,0,3)$ and radius $r=\sqrt{27}$

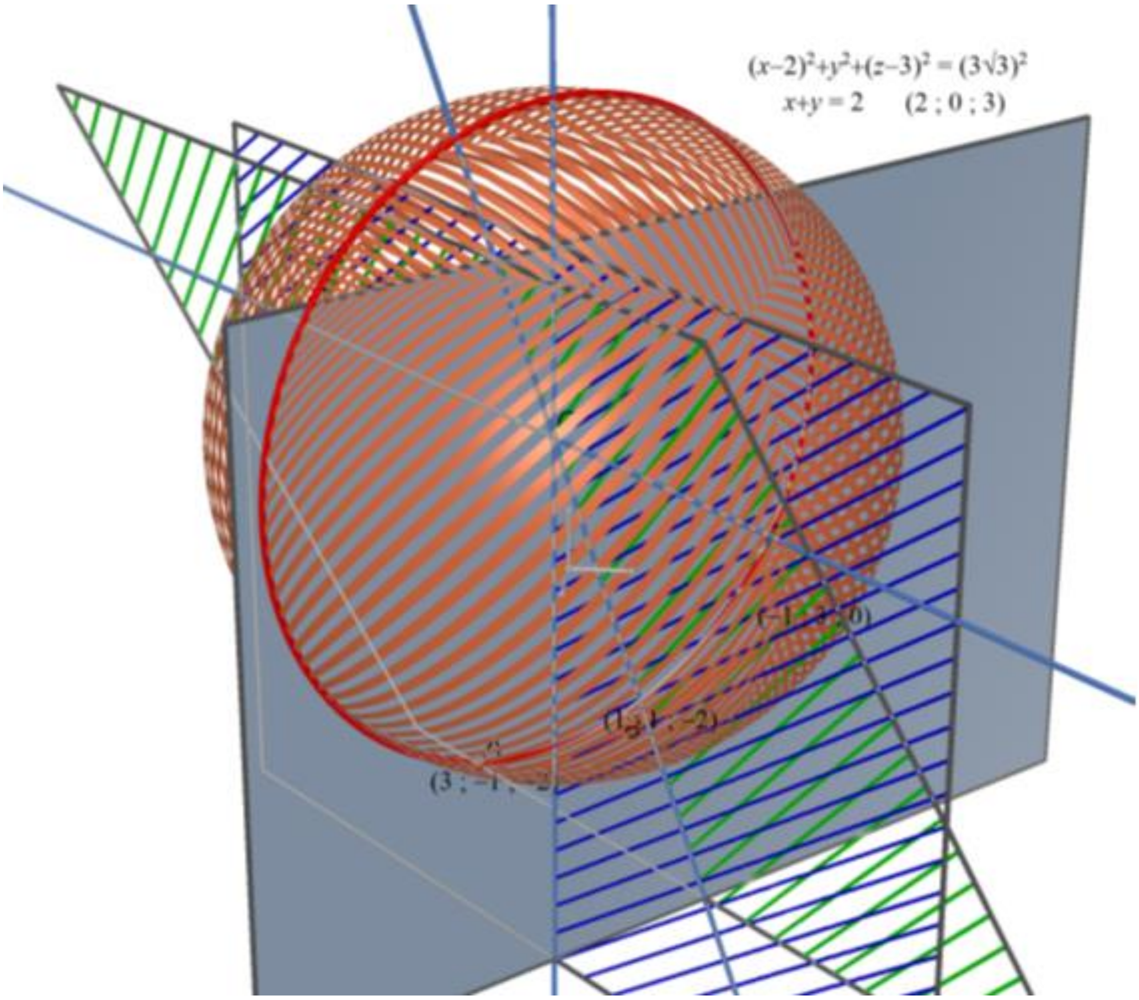
$$\begin{cases} x+y=2 \\ (x-2)^2+y^2+(z-3)^2=27 \end{cases}$$

We open the *Menú Gráfico 3D*. We define and represent the equations of the plane $\pi_{ABC} \equiv x+y-2=0$ and the sphere$(x-2)^2+y^2+(z-3)^2=27$







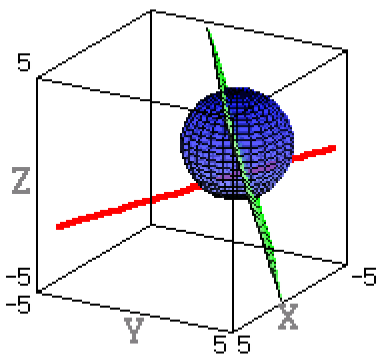


April 15-16: Determine the relative position of the line

$$\mathbf{r} \equiv \begin{cases} \mathbf{x} = 2 - 2\alpha \\ \mathbf{y} = -\dfrac{7}{2} + 3\alpha \\ \mathbf{z} = -2 + \alpha \end{cases}$$

and the sphere

$$\mathbf{E} \equiv \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 + \mathbf{x} - 4\mathbf{y} - 3\mathbf{z} + \frac{1}{2} = 0$$



Solution: Completing squares, the equation of the sphere is:

$$\left(x+\frac{1}{2}\right)^2+(y-2)^2+\left(z-\frac{3}{2}\right)^2=6$$

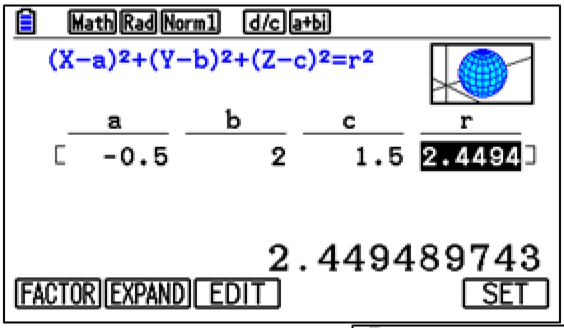
The center of the sphere is the point$O\left(-\frac{1}{2},2,\frac{3}{2}\right)$, the radius is $r=\sqrt{6}$. Let us calculate the projection of the center O on the line r. Let us consider the plane perpendicular to the line r passing through $O\left(-\frac{1}{2},2,\frac{3}{2}\right)$ and characteristic vector, the director of the line $v=(-2,3,1)$. Your equation is:

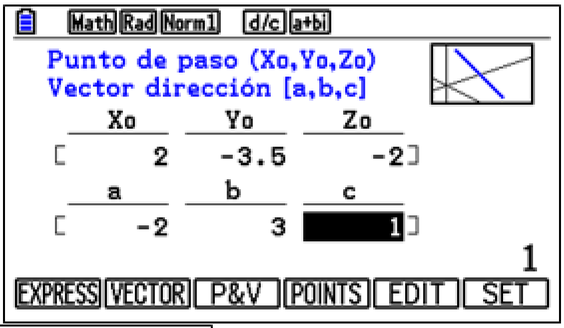
$$\Pi\equiv -2\left(x+\frac{1}{2}\right)+3(y-2)+\left(z-\frac{3}{2}\right)=0$$

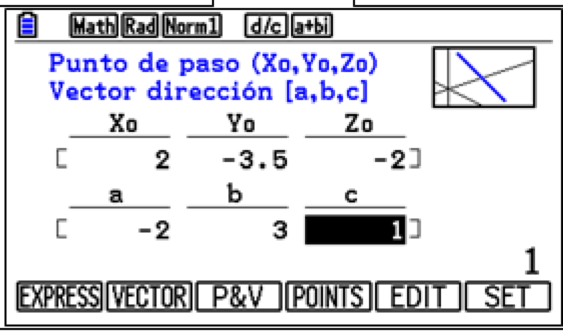
Simplifying:

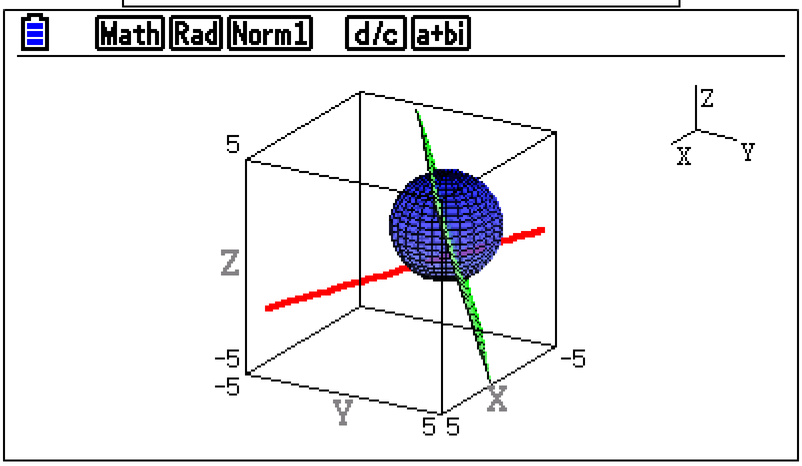
$$\Pi\equiv -2x+3y+z-\frac{17}{2}=0$$

We open the *Menú Gráfico 3D*. We define and represent the sphere, the line and the plane.

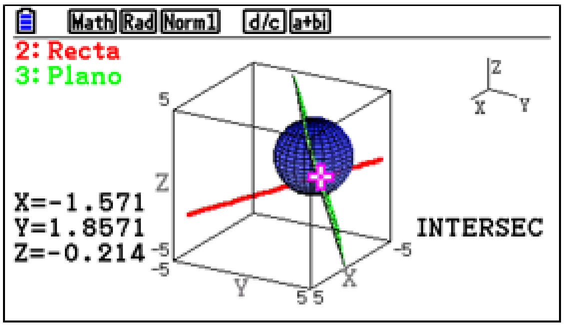






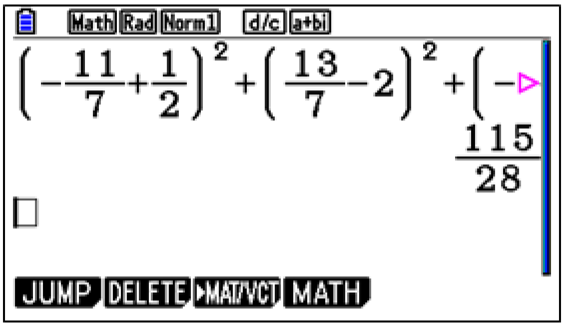


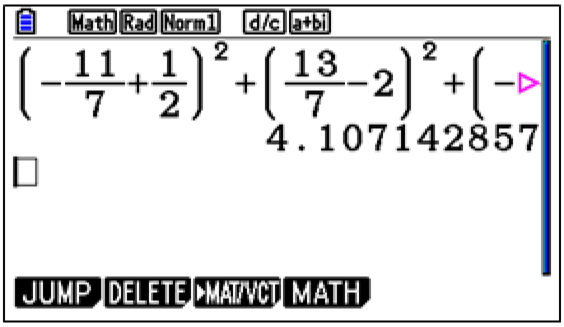
Whit the function *G-Solv,* determine the intersection of the line and the plane (projection point)



The projection point has coordinates $P\left(\frac{-11}{7}, \frac{13}{7}, \frac{-3}{14}\right)$. We calculate the square of the distance between the center O and the projection P.

$$(\mathrm{d}(\mathrm{OP}))^2 = \left(-\frac{11}{7} + \frac{1}{2}\right)^2 + \left(\frac{13}{7} - 2\right)^2 + \left(-\frac{3}{14} - \frac{3}{2}\right)^2$$



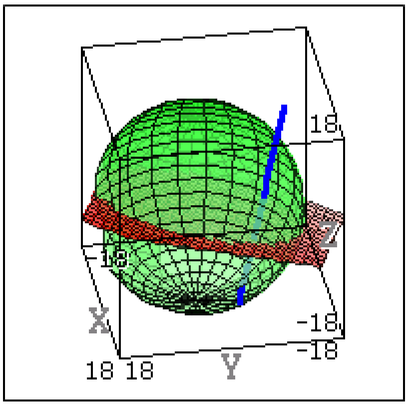


So, the line cuts the sphere since $(\mathrm{d}(\mathrm{OP}))^2 < \mathrm{r}^2 = 6$

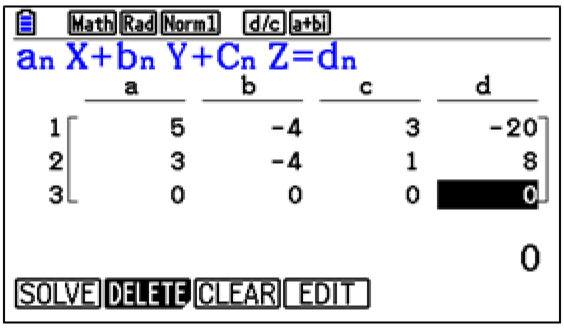
April 18: Determine the equation of the sphere with center O(2, 3, -1) that cuts the line

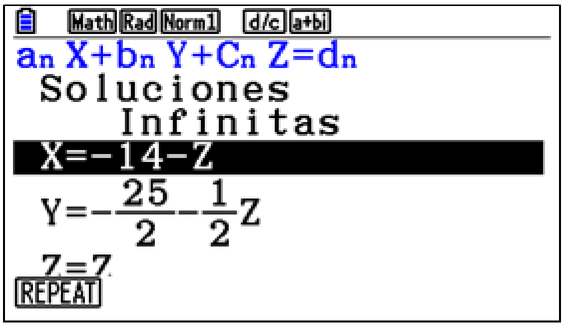
$$s \equiv \begin{cases} 5x - 4y + 3z + 20 = 0 \\ 3x - 4y + z - 8 = 0 \end{cases}$$

on a string of length equal to 16.



Solution: We open the *Menú Ecuación*. We solve the system formed by the line to be determined in parametric form:





The parametric equation of the line is:

$$s \equiv \begin{cases} x = -14 - \alpha \\ y = -\dfrac{25}{2} - \dfrac{1}{2}\alpha \\ z = \alpha \end{cases}$$

A point on the line s is $A\left(-14, -\frac{25}{2}, 0\right)$ and the director vector $v = (-2, -1, 2)$. We determine the projection point of the center O on the line s. The plane that passes through O(2,3,-1) and is perpendicular to the line $s \equiv \begin{cases} x = -14 - \alpha \\ y = -\frac{25}{2} - \frac{1}{2}\alpha \\ z = \alpha \end{cases}$ has characteristic vector the direction vector of the straight line s, $v = (-2, -1, 2)$

The equation is

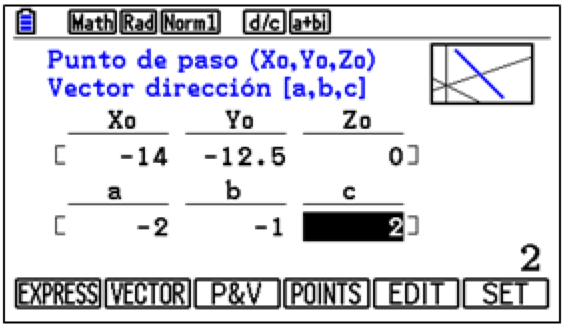
$$\Pi \equiv -2(x - 2) - (y - 3) + 2(z + 1) = 0$$

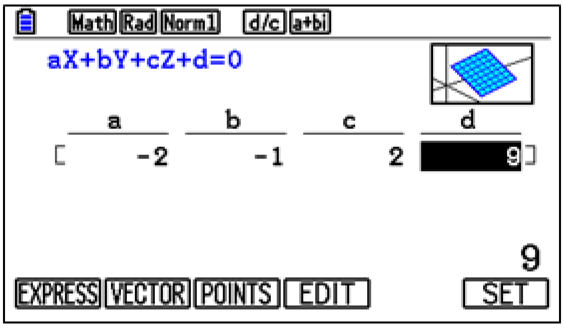
Simplifying:

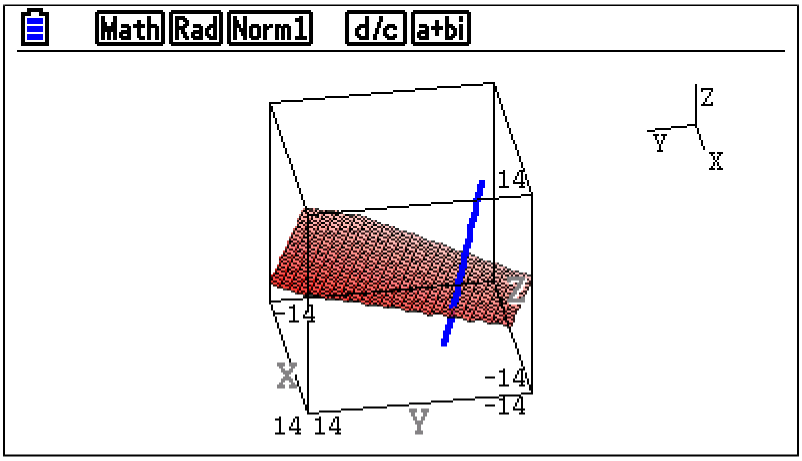
$$\Pi \equiv -2x - y + 2z + 9 = 0$$

We open the *Menú Gráfico 3D.*

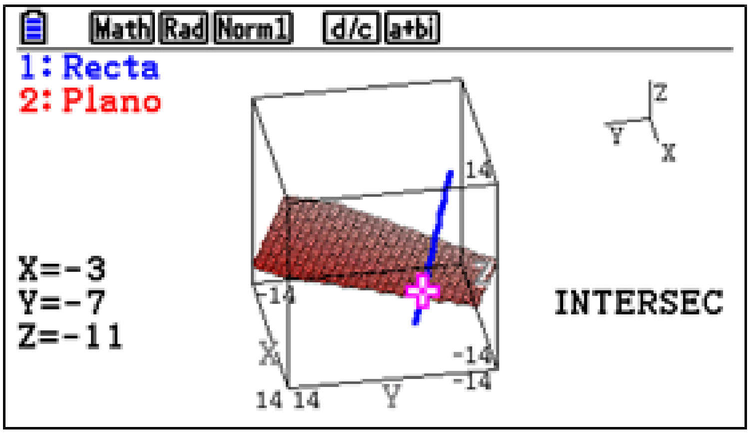
We define and represent the line $s \equiv \begin{cases} x = -14 - \alpha \\ y = -\frac{25}{2} - \frac{1}{2}\alpha \\ z = \alpha \end{cases}$ and the plane $\Pi \equiv -2x - y + 2z + 9 = 0$.







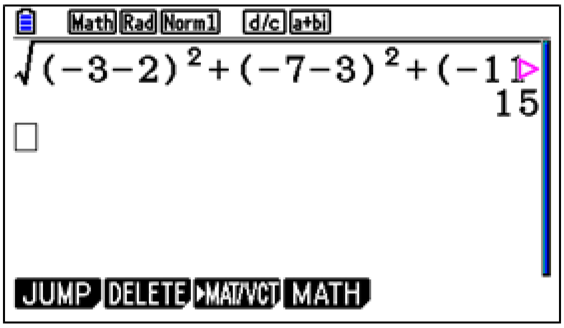
Whit the function *G-Solv,* let's determine the point of intersection, midpoint of the chord.



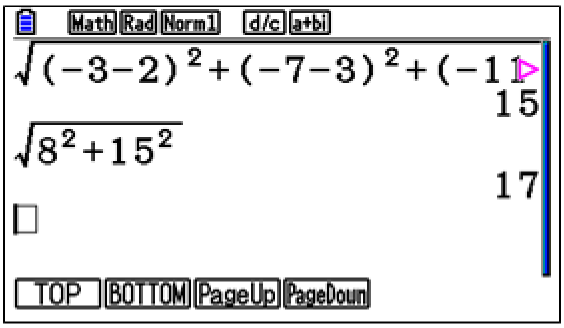
The coordinates of the midpoint of the chord are: $\mathrm{M}(-3,-7,-11)$

We open the *Menú Ejec-Mat*. Let us calculate the distance from the center O to the Point M

$$\mathrm{d}(\mathrm{O},\mathrm{M}) = \sqrt{(-3-2)^2+(-7-3)^2+(-11+1)^2}$$



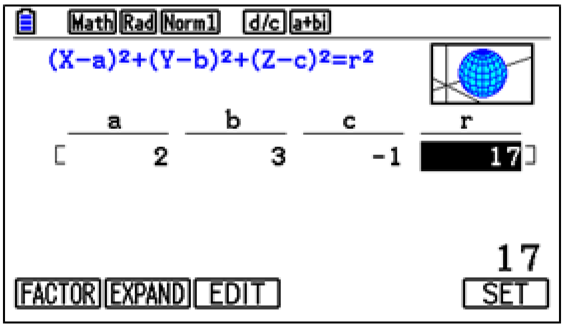
Let r be the radius of the sphere. Applying the Pythagorean theorem: $\mathrm{r}^2 = 8^2 + 15^2$

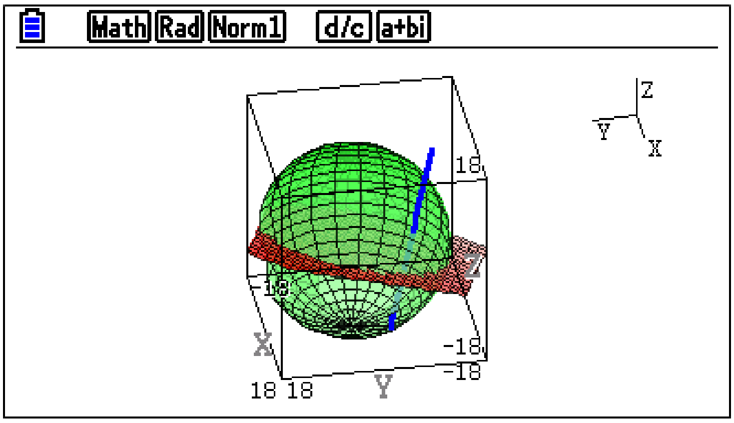


The radius of the sphere is r=17. The equation of the sphere is:

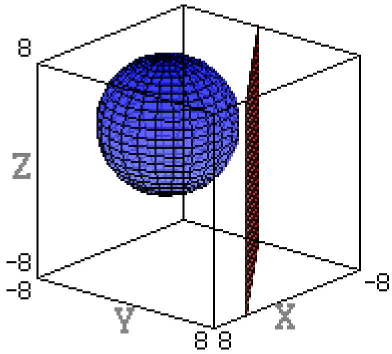
$$(x-2)^2+(y-3)^2+(z+1)^2 = 17^2$$

We open the *Menú Gráfico 3D*. We define and represent the sphere.





April 19-20: In the sphere of equation $E \equiv (x-1)^2 + (y+2)^2 + (x-3)^2 = 25$ determine the point M closest to the plane $\Pi \equiv 3x - 4y + 19 = 0$ and calculate the distance from point M to this plane.

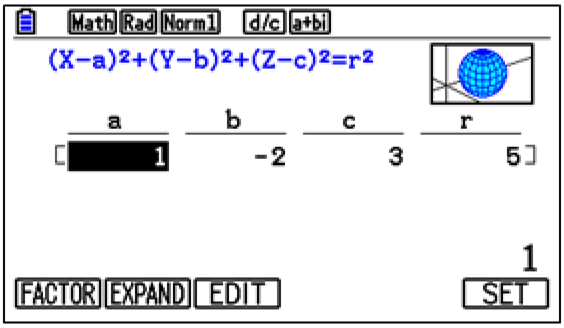


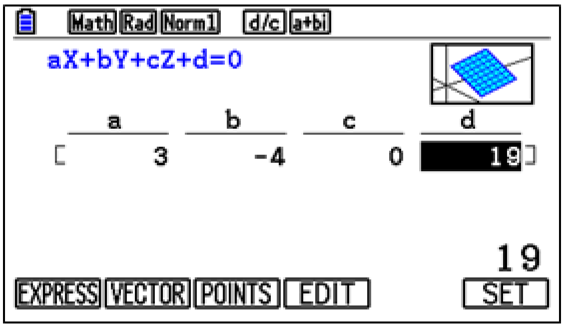
Solution: The sphere has as its center the Point $O(1,-2,3)$ and radius $r = 5$. Let us see that the plane is exterior to the sphere. Let's calculate the distance from the center $O(1,-2,3)$ to the plane $\Pi \equiv 3x - 4y + 19 = 0$

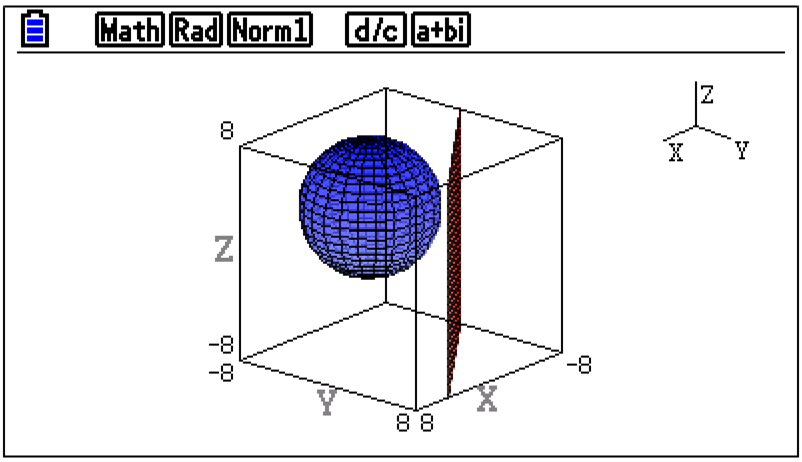
$$d(O,\Pi) = \left|\frac{3\cdot 1 - 4\cdot(-2) + 0\cdot 3 + 19}{\sqrt{3^2+(-4)^2+0^2}}\right| = 6, \quad d(O,\Pi) = 6 > r = 5$$

So the plane is exterior to the sphere.

We open the *Menú Gráfico* 3D. We define and represent the sphere and the plane.







The line perpendicular to the plane $\Pi \equiv 3x - 4y + 19 = 0$ what goes through the center $O(1, -2, 3)$ has the characteristic director vector of the plane $a = (3, -4, 0)$. Your equation is:

$$r \equiv (x, y, z) = (1, -2, 3) + \alpha(3, -4, 0)$$

Any point on the line has coordinates:

$$M(1 + 3\alpha, -2 - 4\alpha, 3)$$

Substituting the coordinates of the point in the equation of the plane:

$$(3\alpha)^2 + (-4\alpha)^2 = 25$$

Solving the equation:

$$\alpha = 1, -1$$

If $\alpha = 1$, the point of intersection has coordinates $M_1(4, -6, 3)$. If $\alpha = -1$, the point of intersection has coordinates $M_2(-2, 2, 3)$

Let us calculate the distance of the point $M_1(4, -6, 3)$ to the plane $\Pi \equiv 3x - 4y + 19 = 0$

$$d(M_1, \Pi) = \left|\frac{3 \cdot 4 - 4 \cdot (-6) + 0 \cdot 3 + 19}{\sqrt{3^2 + (-4)^2 + 0^2}}\right| = 11$$

Let us calculate the distance of the point $M_2(-2, 2, 3)$ to the plane $\Pi \equiv 3x - 4y + 19 = 0$

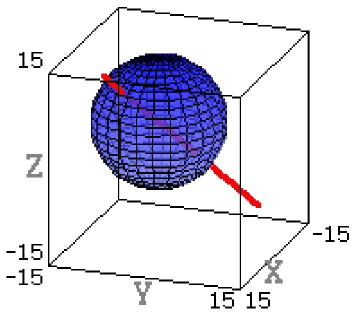
$$d(M_2, \Pi) = \left|\frac{3 \cdot (-2) - 4 \cdot 2 + 0 \cdot 3 + 19}{\sqrt{3^2 + (-4)^2 + 0^2}}\right| = 1$$

The closest point to the plane $\Pi \equiv 3x - 4y + 19 = 0$ is $M_2(-2, 2, 3)$

April 21-22: Calculate the shortest distance from point A (1, -1, 3) to the sphere

$$\mathbf{E \equiv x^2 + y^2 + z^2 - 6x + 4y - 10z - 62 = 0}$$

At what point on the sphere is the shortest distance achieved?



Solution: Completing squares:

$$E \equiv (x - 3)^2 + (y + 2)^2 + (z - 5)^2 = 62 + 9 + 4 + 25 = 10^2$$

The coordinates of the center of the sphere is O(3, -2, 5) and the radius $R = 10$

Let us study the relative position of the point A(1, -1, 3) with respect to the sphere.

$$(1 - 3)^2 + (-1 + 2)^2 + (3 - 5)^2 = 9 < 10^2$$

The point A(1, -1, 3) is inside the sphere.

$$\overline{OA} = \sqrt{(1 - 3)^2 + (-1 + 2)^2 + (3 - 5)^2} = 3$$

The shortest distance is:

$$d_{mín} = R - \overline{OA} = 10 - 3 = 7$$

The longest distance is:

$$d_{màx} = R + \overline{OA} = 10 + 3 = 13, \qquad \overrightarrow{OA} = (-2, 1, -2)$$

The equation of the line passing through the points O and A is:

$$r_{OA} \equiv (x, y, z) = (3, -2, 5) + \alpha(-2, 1, -2)$$

Any point on the line $r_{OA} \equiv (x, y, z) = (3, -2, 5) + \alpha(-2, 1, -2)$ has coordinates:

$$P(3 - 2\alpha, -2 + \alpha, 5 - 2\alpha)$$

Let us determine the intersection of the line and the sphere. We substitute the coordinates of the point $P(3 - 2\alpha, -2 + \alpha, 5 - 2\alpha)$ in the equation of the sphere:

$$(-2\alpha)^2 + \alpha^2 + (-2\alpha)^2 = 100$$

Solving the equation:

$$\alpha = \frac{10}{3}, -\frac{10}{3}$$

If $\alpha = \frac{10}{3}$

$$P_1\left(\frac{-11}{3}, \frac{4}{3}, \frac{-5}{3}\right), \qquad \overline{AP_1} = \sqrt{\left(\frac{-11}{3} - 1\right)^2 + \left(\frac{4}{3} + 1\right)^2 + \left(\frac{-5}{3} - 3\right)^2} = 7$$

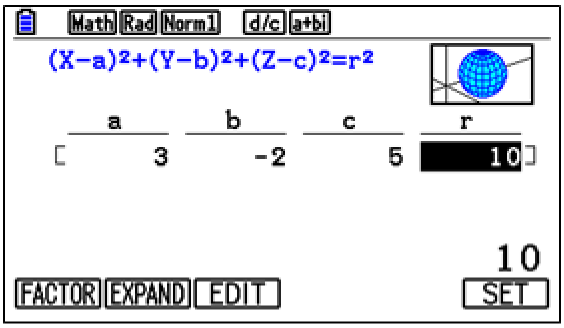
$P_1\left(\frac{-17}{3}, \frac{7}{3}, \frac{11}{3}\right)$ is the closest point of A.

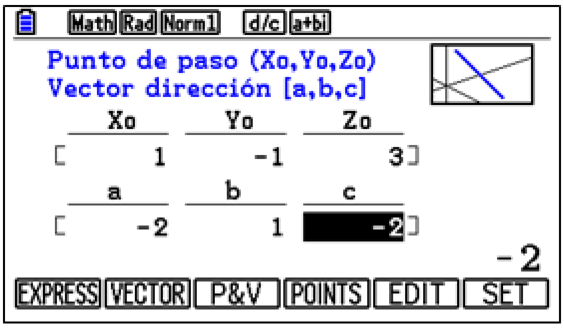
If $\alpha = -\frac{10}{3}$

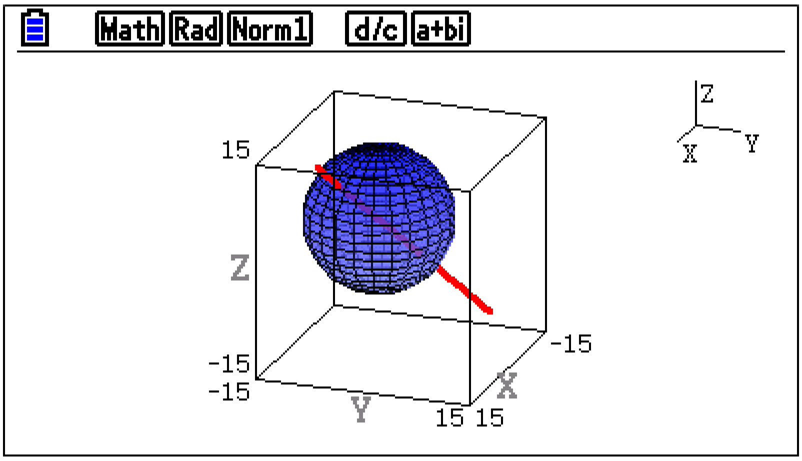
$$P_2\left(\frac{29}{3}, \frac{-16}{3}, \frac{35}{3}\right), \qquad \overline{AP_2} = \sqrt{\left(\frac{29}{3} - 1\right)^2 + \left(\frac{-16}{3} + 1\right)^2 + \left(\frac{35}{3} - 3\right)^2} = 13$$

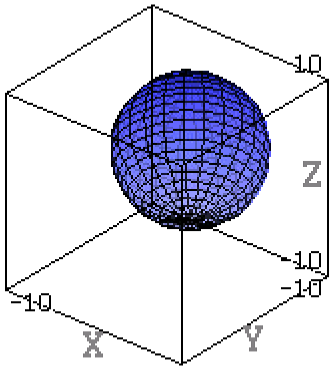
$P_2\left(\frac{29}{3}, \frac{-16}{3}, \frac{35}{3}\right)$ is the furthest point from A.

We open the *Menú Gráfico 3D*. We define and represent the sphere and the line









April 23: Determine the equation of the sphere passing through the points A(3, 1, -3); B(-2, 4, 1); C(-5, 0, 0) and has center in the plane

$$\boldsymbol{\Pi \equiv 2x + y - z + 3 = 0}$$

Solution: Let's calculate the components of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$

$$\overrightarrow{AB} = (-5, 3, 4), \quad \overrightarrow{AC} = (-8, -1, 3)$$

Vectors are linearly independent since the components are not proportional.

$$\frac{-5}{-8} \neq \frac{3}{-1}$$

Let's calculate the midpoints of the segments $\overline{AB}, \overline{AC}$

The midpoint of the segment $\overline{AB}$ is $D\left(\frac{1}{2}, \frac{5}{2}, -1\right)$

The midpoint of the segment $\overline{AC}$ is $E\left(-1, \frac{1}{2}, \frac{-3}{2}\right)$

The median plane of the segment $\overline{AB}$ has equation:

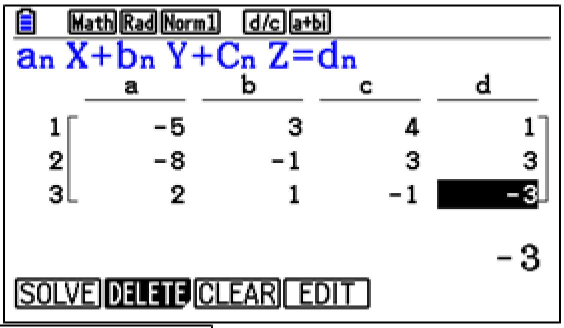
$$\Pi_1 \equiv -5\left(x - \frac{1}{2}\right) + 3\left(y - \frac{5}{2}\right) + 4(z + 1) = 0, \qquad \Pi_1 \equiv -5x + 3y + 4z = 1$$

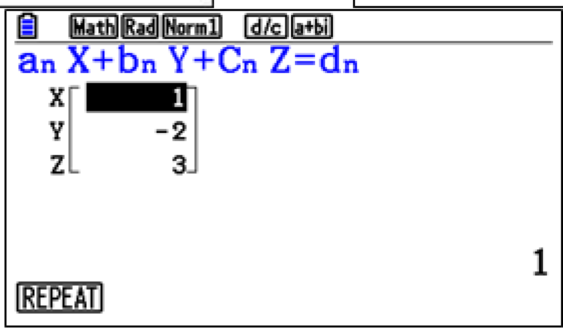
The median plane of the segment $\overline{AC}$ has equation:

$$\Pi_2 \equiv -8(x + 1) - \left(y - \frac{1}{2}\right) + 3\left(z + \frac{3}{2}\right) = 0, \quad \Pi_2 \equiv -8x - y + 3z = 3$$

The center of the sphere is equal to the intersection point of the three planes.

We open the *Menú Ecuación*:



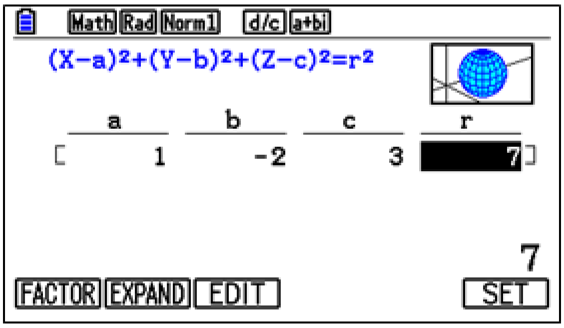


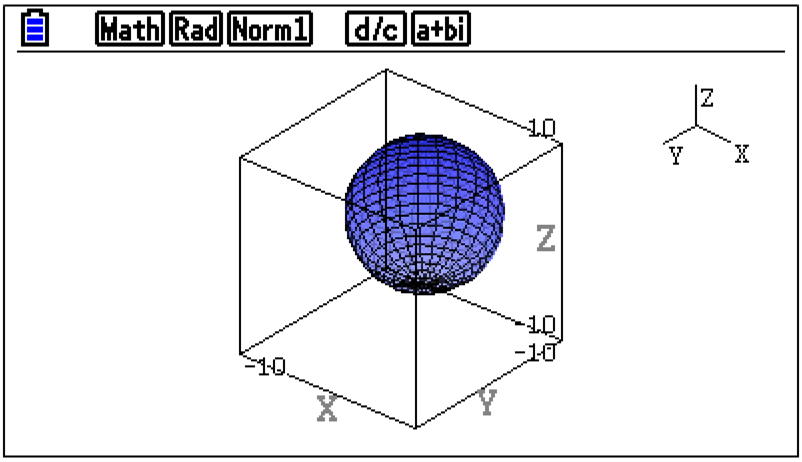
The center of the sphere is the point O(1, -2, 3). The radius of the sphere is:

$$r = \overline{OA} = \sqrt{(3-1)^2 + (1+2)^2 + (-3-3)^2} = 7$$

The radius of the sphere is $r = 7$. The equation of the sphere is $(x-1)^2 + (y+2)^2 + (z-3)^2 = 7^2$

We open the *Menú Gráfico 3D*. We define and represent the sphere.





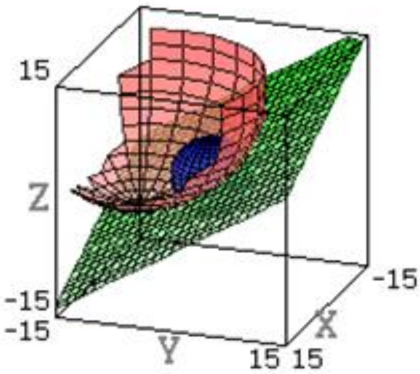
April 25-26: Let be the spheres of equations

$$\mathbf{E_1 \equiv x^2 + y^2 + z^2 = 25,}$$
$$\mathbf{E_2 \equiv x^2 + y^2 + z^2 - 10x + 15y - 25z = 0}$$

Prove that the two spheres are secant.

Determine the plane that contains the intersection of the two spheres.

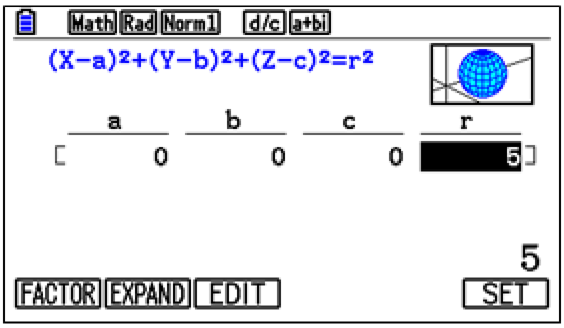
Find the center and radius of the intersecting circle.

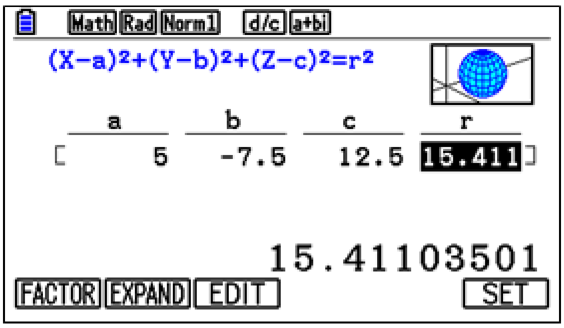


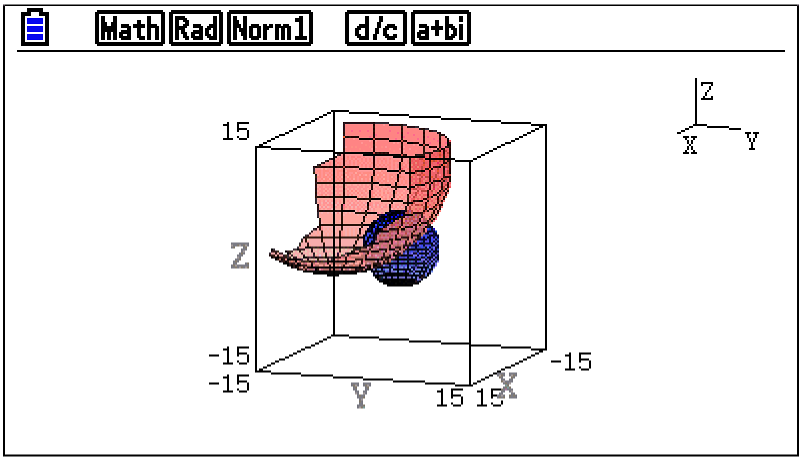
Solution: The sphere $E_1 \equiv x^2 + y^2 + z^2 = 25$ has center $O_1(0,0,0)$ and radius $R_1 = 5$. Completing squares on the sphere $E_2 \equiv x^2 + y^2 + z^2 - 10x + 15y - 25z = 0$

$$(x-5)^2 + \left(y + \frac{15}{2}\right)^2 + \left(z - \frac{25}{2}\right)^2 = 25 + \frac{225}{4} + \frac{625}{4} = \left(\frac{5}{2}\sqrt{38}\right)^2$$

The center of the sphere is $O_2\left(5, -\frac{15}{2}, \frac{25}{2}\right)$ and radius $R_2 = \frac{5}{2}\sqrt{38}$. We open the *Menú Gráfico 3D*. We define and represent the two spheres.







Let us see analytically that the two spheres are secant. Let us calculate the distance between the two centers.

$$\overline{O_1O_2} = \sqrt{(5-0)^2 + \left(-\frac{15}{2} - 0\right)^2 + \left(\frac{25}{2} - 0\right)^2} = \frac{5}{2}\sqrt{38}$$

The sum of the radii is:

$$R_1 + R_2 = 5 + \tfrac{5}{2}\sqrt{38} > \overline{O_1O_2}$$

The difference of the radii is:

$$R_2 - R_1 = \frac{5}{2}\sqrt{38} - 5 > \overline{O_1O_2}$$

So the two spheres are secant.

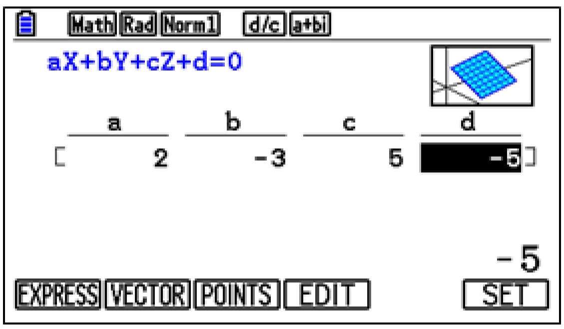
The intersection plane of the two spheres is the plane formed by the difference of the equations of the two spheres:

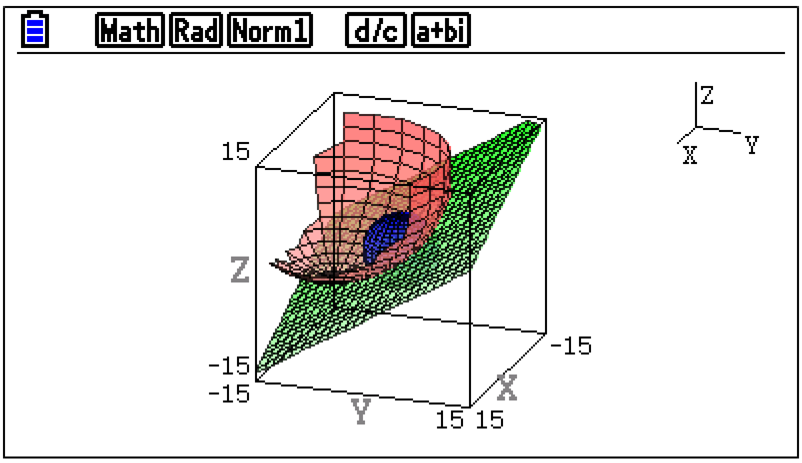
$$\Pi \equiv 10x - 15y + 25z = 25$$

Simplifying:

$$\Pi \equiv 2x - 3y + 5z - 5 = 0$$

Let us represent the plane.





The center of the intersection circle of the two spheres is the projection of the center $O_1(0,0,0)$ on the plane $\Pi \equiv 2x - 3y + 5z - 5 = 0$. The straight line r perpendicular to the plane that passes through $O_1(0,0,0)$ has the characteristic direction vector of the plane a=(2, -3, 5) and its equation is:

$$r \equiv (x, y, z) = \alpha(2, -3, 5)$$

We substitute the coordinates of any point on the line r, $(2\alpha, -3\alpha, 5\alpha)$ in the equation of the plane:

$$2(2\alpha) - 3(-3\alpha) + 5(5\alpha) - 5 = 0$$

Solving the equation:

$$\alpha = \frac{5}{38}$$

The intersection of the plane and the line is:

$$O\left(\frac{5}{19}, -\frac{15}{38}, \frac{25}{38}\right), \quad \overline{O_1O} = d(O_1, \Pi) = \left|\frac{-5}{\sqrt{38}}\right| = \frac{5}{\sqrt{38}}$$

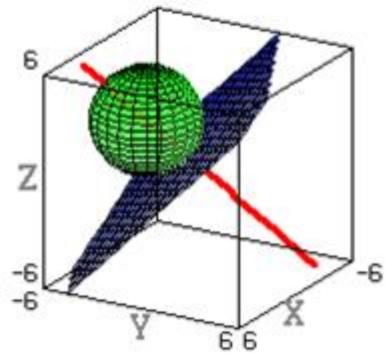
Consider the right triangle formed by the legs $\overline{O_1O}$, O and a point of the circumference and hypotenuse $R_1 = 5$. Let R be the radius of the circle. Applying the Pythagorean theorem:

$$5^2 = R^2 + \left(\frac{5}{\sqrt{38}}\right)^2 \quad R = 5 \cdot \sqrt{\frac{37}{38}} \approx 4.93$$

April 27: Prove that the point T(1, 0, 1) belongs to the plane

$$\boldsymbol{\pi \equiv x - 2y + 2z = 3.}$$

Determine the equation of the sphere that passes through the point P(1, 0, 5) and is tangent at T to the plane π.



Solution: We substitute the point T(1,0,1) in the equation of the plane $\pi \equiv x - 2y + 2z = 3$ and let's see that it becomes an equality:

$$1 - 2 \cdot 0 + 2 \cdot 1 = 3$$

The center belongs to the line r perpendicular to the plane passing through the point T(1, 0, 1). The direction vector of the line r is the characteristic of the plane π, $a = (1, -2, 2)$

$$r \equiv (x, y, z) = (1, 0, 1) + \mu(1, -2, 2)$$

The center of the sphere has coordinates:

$$O(1 + \mu, -2\mu, 1 + 2\mu)$$

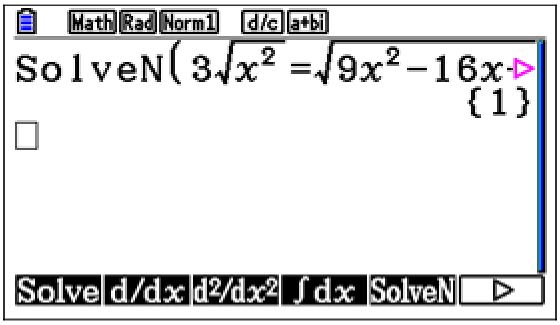
The center fulfills $d(O, T) = d(O, P) = R$, sphere radius.

$$d(O, T) = \sqrt{(-\mu)^2 + (2\mu)^2 + (-2\mu)^2} = 3\sqrt{\mu^2}$$

$$d(O, P) = \sqrt{(-\mu)^2 + (2\mu)^2 + (4 - 2\mu)^2} = \sqrt{9\mu^2 - 16\mu + 16}$$

$$3\sqrt{\mu^2} = \sqrt{9\mu^2 - 16\mu + 16}$$

We open the *Menú Ejec-Mat*: We solve the equation:



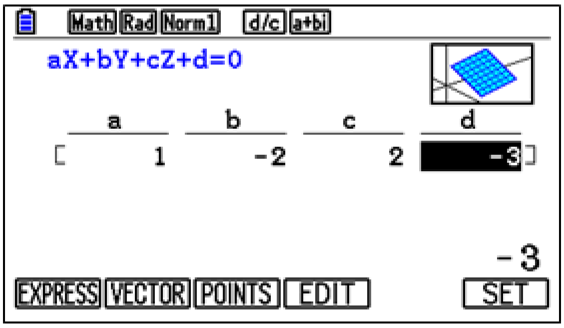
The solution is:

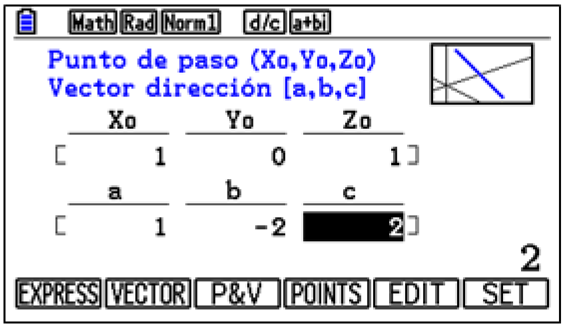
$$\mu = 1$$

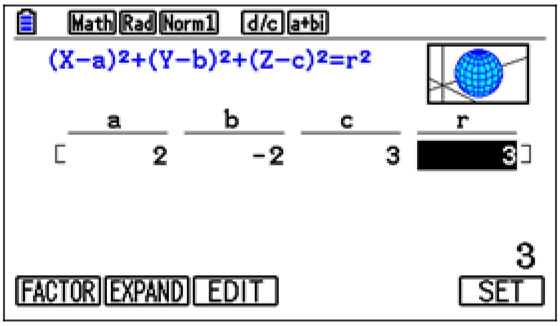
The center of the sphere is O(2, -2, 3). The radius is R = 3·1 = 3. The equation of the sphere is:

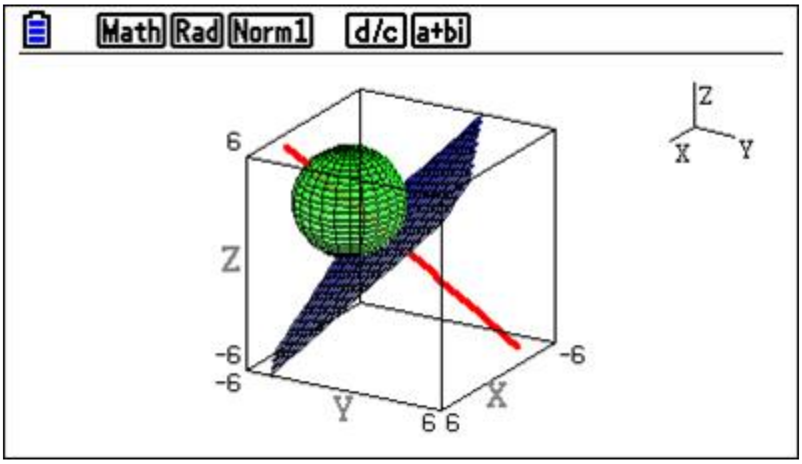
$$(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 3^2$$

We open the *Menú Gráfico 3D*. We define and represent the plane $\pi \equiv x - 2y + 2z = 3$, the line $r \equiv (x, y, z) = (1, 0, 1) + \mu(1, -2, 2)$ and the sphere $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 3^2$





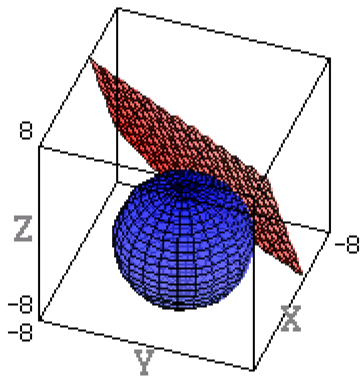




April 28: Determine the equation of the plane tangent to the sphere

$$(x - 3)^2 + (y - 1)^2 + (z + 2)^2 = 24$$

pass through the point M(-1, 3, 0)



Solution: The sphere $(x-3)^2+(y-1)^2+(z+2)^2=24$ have centre $O(3,1,-2)$ and radius $r=\sqrt{24}$. The point M(-1, 3, 0) belongs to the sphere since $(-1-3)^2+(3-1)^2+(0+2)^2=24$. The tangent plane to the sphere passes through the point M(-1, 3, 0) and has characteristic vector

$$\overrightarrow{OM}=(-4,2,2)$$

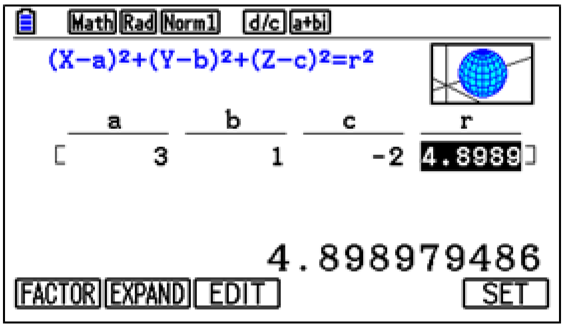
The equation of the plane is:

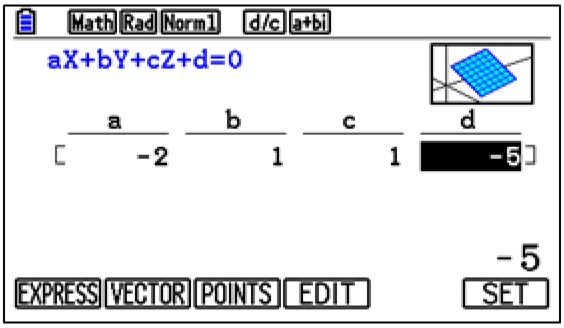
$$-4(x+1)+2(y-3)+2(z-0)=0$$

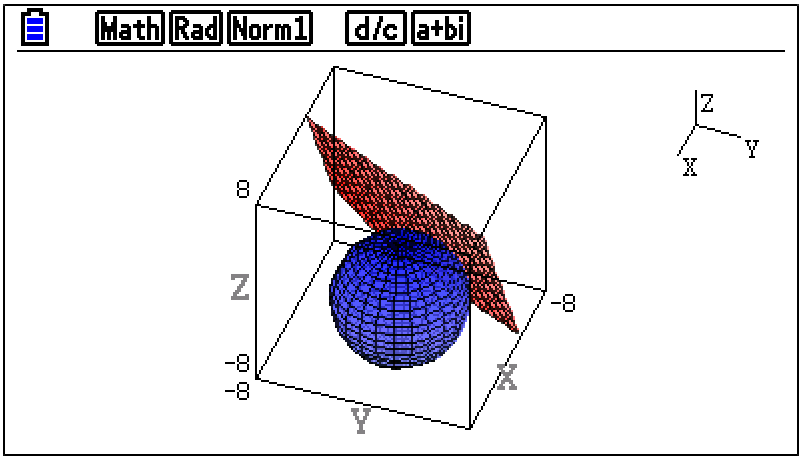
Simplifying:

$$-2x+y+z-5=0$$

We open the *Menú Gráfico 3D*: We define and represent the sphere and the plane.



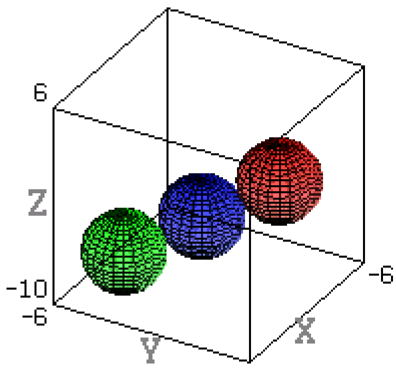




April 29-30: Let the sphere

$$\mathbf{x^2+y^2+z^2-6x-4y+8z+20=0}$$

Calculate the sphere of equal radius, exterior tangent at the point A(1, 4, -3) of the sphere. Calculate the sphere of equal radius, exterior tangent at the point diametrically opposite point A of the sphere.



Solution: Completing squares:

$$E\equiv(x-3)^2+(y-2)^2+(z+4)^2=3^2$$

Sphere E has center O(3, 2, -4) and radius r = 3. Note that point A(1, 4, -3) belongs to sphere E since it satisfies its equation:

$$E\equiv(1-3)^2+(4-2)^2+(-3+4)^2=3^2$$

The centre $O_1$of the tangent sphere at the Point $A(1,4,-3)$ satisfy: $\overrightarrow{OA} = \overrightarrow{AO_1}$

$$(-2,2,1) = (x-2, y-4, z+3)$$

Solving the equation: $O_1(-1,6,-2)$. The equation of the sphere of radius 3, tangent at A to sphere E has equation:

$$E_1 \equiv (x+1)^2 + (y-6)^2 + (z+2)^2 = 3^2$$

The Point A' diametrically opposite point A, satisfies: $\overrightarrow{AO} = \overrightarrow{OA'}$

$$(2,-2,-1) = (x-3, y-2, z+4)$$

Solving the equation: $A'(5,0,-5)$.

The centre O_2 of the tangent sphere at the point $A'(5,0,-5)$ satisfy:

$$\overrightarrow{O_1O} = \overrightarrow{OO_2}, \quad (4,-4,-2) = (x-3, y-2, z+4)$$

Solving the equation: $O_2(7,-2,-6)$

The equation of the sphere of radius 3 tangent at A' to the sphere E has equation:

We open the *Menú Gráfico 3D.* We define and represent the three spheres.