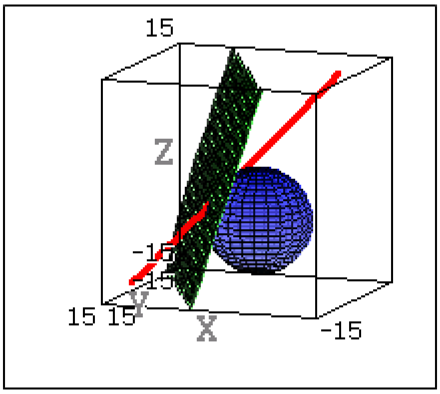
SOLUTIONS JULY 2022

PROBLEMS NOT TO LOSE "THE TOUCH". AUTHOR: RICARD PEIRÓ i ESTRUCH. IES "Abatos". Valencia



July 1-2: Through the points of intersection of the line

$$r \equiv \begin{cases} x = -5 + 3t \\ y = -11 + 5t \\ z = 9 - 4t \end{cases}$$

and from the sphere of equation:

$$E \equiv (x+2)^2 + (y-1)^2 + (z+5)^2 = 49$$

tangent planes have been drawn. Determine your equations

Solution: The centre of the sphere is O (-2, 1, -5) and the radius R = 7. We determine the intersection points of the line and the sphere. Any point on the line

$$r \equiv \begin{cases} x = -5 + 3t \\ y = -11 + 5t \\ z = 9 - 4t \end{cases}$$

is P(5+3t, -11+5t, 9-4t). We substitute its coordinates in the equation of the sphere:

$$(-3 + 3t)^2 + (-12 + 5t)^2 + (14 - 4t)^2 = 49$$

Simplifying:

$$t^2 - 5t + 6 = 0$$

Solving the equation: t = 2, 3. The coordinates of the intersection points are:

$$P_1(1, -1, 1), \quad P_2(4, 4, -3)$$

The characteristic vector of the tangent plane to the sphere at the point $P_1(1, -1, 1)$ is:

$$\overrightarrow{OP_1} = (3, -2, 6)$$

The equation of the plane is:

$$\pi_1 \equiv 3(x-1) - 2(y+1) + 6(z-1) = 0$$

Simplifying:

$$\pi_1 \equiv 3x - 2y + 6z - 11 = 0$$

The characteristic vector of the tangent plane to the sphere at the point $P_2(4, 4, -3)$ is:

$$\overrightarrow{OP_2} = (6, 3, 2)$$

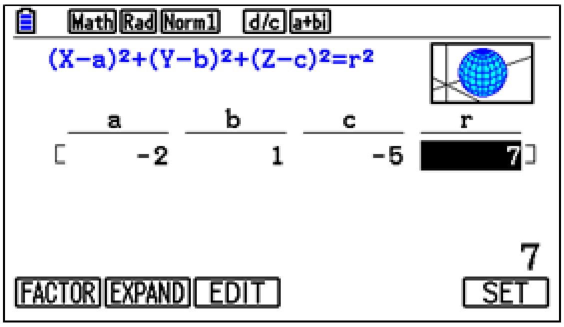
The equation of the plane is:

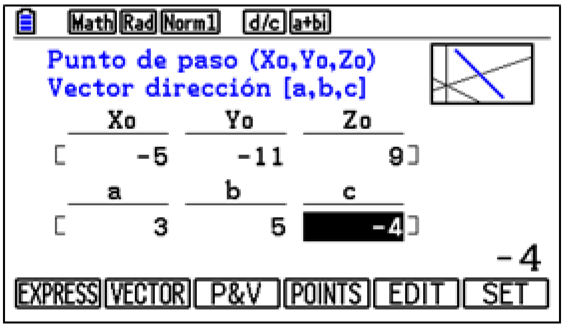
$$\pi_2 \equiv 6(x-4) + 3(y-4) + 2(z+3) = 0$$

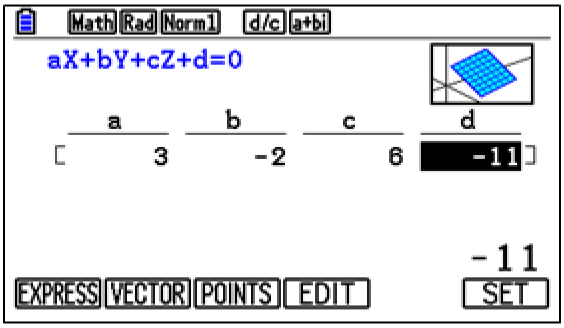
Simplifying:

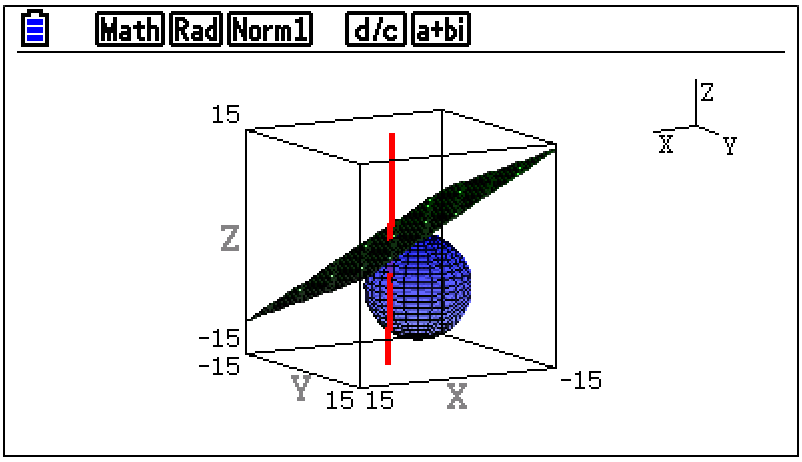
$$\pi_2 \equiv 6x + 3y + 2z - 30 = 0$$

We open the *Menú Gráfico 3D*. We define the equations of the sphere, the line and the plane $\pi_1 \equiv 3x - 2y + 6z - 11 = 0$



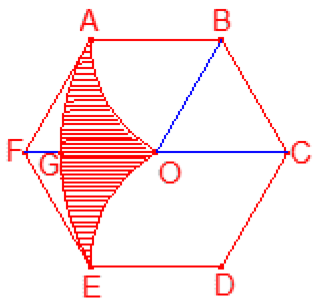






July 4-5: Let ABCDEF be the regular hexagon with centre O and side c. From points B and D as centres and with radius c two arcs of circumference are drawn. From point C as centre is draw the arc $\overset{\frown}{AGE}$.

Determine the area of the shaded area.

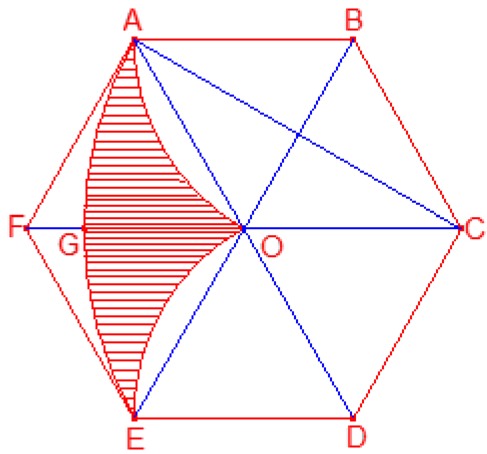


Solution: We will have:

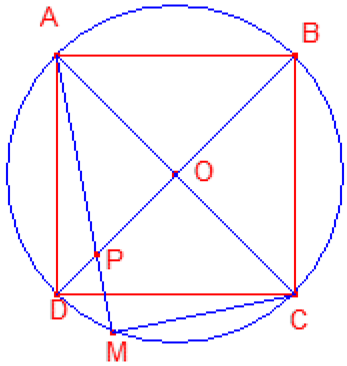
$$\overline{AE} = \sqrt{3}c$$

The area of one of the large lunules is equal to the area of the 60° sector and radius $\sqrt{3}c$ minus the area of two 60° circular sectors of radius c:

$$S_{shaded} = \frac{1}{6}\pi\left(\sqrt{3}c\right)^2 - 2\left(\frac{1}{6}\pi c^2\right) = \frac{\pi}{6}c^2.$$



<u>July 6-13:</u> Square ABCD is inscribed in a circle of radius 30. The chord $\overline{AM}$ measure 50 and cut the diagonal $\overline{BD}$ at point P.
Determine the length of the segment $\overline{AP}$.



<u>Solution 1:</u> Applying the power of point P with respect to the circumference:

$$\overline{AP}\cdot\overline{MP}=\overline{DP}\cdot\overline{BP}. \qquad \overline{AP}\cdot\left(30-\overline{AP}\right)=\overline{DP}\cdot\left(60-\overline{DP}\right). \qquad \overline{AP}\cdot\left(30-\overline{AP}\right)=-\overline{DP}^2+60\overline{DP} \qquad (1)$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ABD}$:

$$\overline{AD}=30\sqrt{2}$$

Applying the law of cosines to the triangle $\overset{\Delta}{ADP}$:

$$\overline{AP}^2=\left(30\sqrt{2}\right)^2+\overline{DP}^2-2\left(30\sqrt{2}\right)\overline{DP}\frac{\sqrt{2}}{2}. \qquad \overline{AP}^2=\left(30\sqrt{2}\right)^2+\overline{DP}^2-60\overline{DP} \qquad (2)$$

Adding the expressions (1) and (2):

$$\overline{AP}\cdot\left(30-\overline{AP}\right)+\overline{AP}^2=1800. \qquad 30\overline{AP}=1800. \qquad \overline{AP}=36.$$

<u>Solution 2:</u> We will have:

$$\overline{OA}=30, \ \overline{AC}=60.$$

Let O be the centre of the square. Right triangles $\overset{\Delta}{APO}$, $\overset{\Delta}{ACM}$ are similar. Applying Thales' theorem:

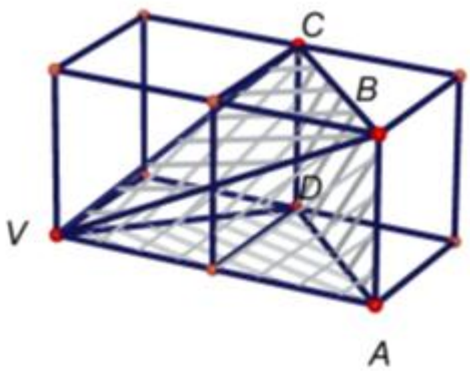
$$\frac{\overline{AP}}{\overline{AC}}=\frac{\overline{OA}}{\overline{AM}}. \qquad \frac{\overline{AP}}{60}=\frac{30}{50}.$$

Solving the equation:

$$\overline{AP}=36$$

<u>July 7:</u> Let them be two equal cubes joined by a common face (see figure).

Determine the ratio between the volume of the pyramid ABCDV and the sum of the volumes of the two cubes.



<u>Solution:</u> Let $\overline{CD}=a$, be the edge of the two cubes. The sum of the volumes of the two cubes is:

$$V_{2c}=2a^3$$

The rectangle ABCD is the base of the pyramid ABCDV.

$$\overline{VC} = a\sqrt{3}, \overline{CD} = a, \overline{VD} = a\sqrt{2}$$

Applying the inverse Pythagorean theorem, the triangle $\overset{\Delta}{VDC}$ it is rectangle $\angle VDC = 90°$

$$\overline{VD} = a\sqrt{2}, \overline{AD} = a\sqrt{2}, \overline{VA} = 2a$$

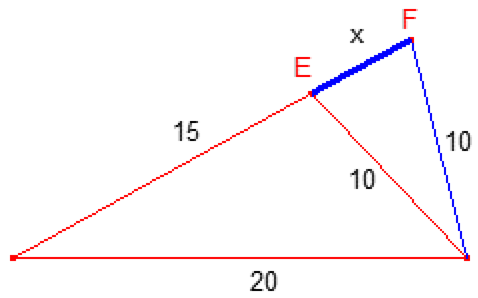
Applying the inverse Pythagorean theorem, the triangle $\overset{\Delta}{VDA}$ is right $\angle VDA = 90°$. So, $\overline{VD}$ is perpendicular to the base, height of the pyramid. The volume of the pyramid is:

$$V_{ABCDV} = \frac{1}{3}\overline{DA} \cdot \overline{DC} \cdot \overline{VD} = \frac{1}{3}a\sqrt{2} \cdot a \cdot a\sqrt{2} = \frac{2}{3}a^3$$

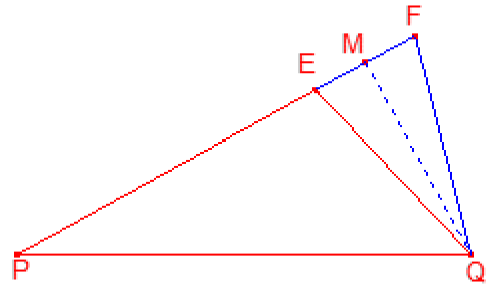
The ratio between the volumes is:

$$\frac{V_{ABCDV}}{V_{2c}} = \frac{\frac{2}{3}a^3}{2a^3} = \frac{1}{3}$$

July 8: In the figure, calculate the measure of the segment $\overline{EF}$.



Solution: Be $\overline{PQ} = 20$, $\overline{PE} = 15$, $\overline{QE} = \overline{QF} = 10$. The triangle $\overset{\Delta}{QEF}$ is isosceles. Let M be the midpoint of the segment $\overline{EF}$. $\angle EMQ = 90^o$. Be $\overline{QM} = y$. Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{PMQ}$:



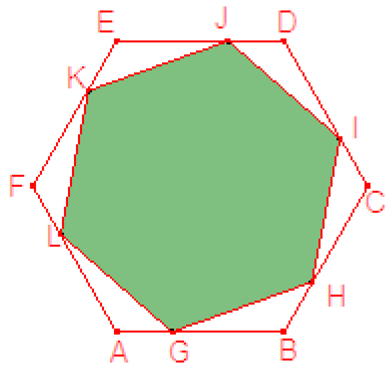
$$20^2 = y^2 + \left(15 + \frac{x}{2}\right)^2.$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{EMQ}$:

$$10^2 = y^2 + \left(\frac{x}{2}\right)^2.$$

Solving the system formed by the two expressions:

$$x = 5.$$



Julio 9-16: In a regular hexagon ABCDEF the regular hexagon GHIJKL has been inscribed such that $\overline{AG} = \frac{1}{3}\overline{AB}$.

Calculate the ratio between the areas of the two hexagons.

Solution: Let $\overline{AB} = c$ be the side of hexagon ABCDEF.

$$\overline{AG} = \frac{1}{3}c,\ \overline{AL} = \frac{2}{3}c.$$

$\angle FAG = 120^o$, interior angle of the regular hexagon. Applying the law of cosines to the triangle $\overset{\Delta}{LAG}$:

$$\overline{LG}^2 = \left(\frac{1}{3}c\right)^2 + \left(\frac{2}{3}c\right)^2 - 2\cdot\frac{1}{3}\cdot\frac{2}{3}\left(\frac{-1}{2}\right) = \frac{7}{9}c^2.$$

The ratio of the areas of two regular hexagons is equal to the square of the ratio of the sides.

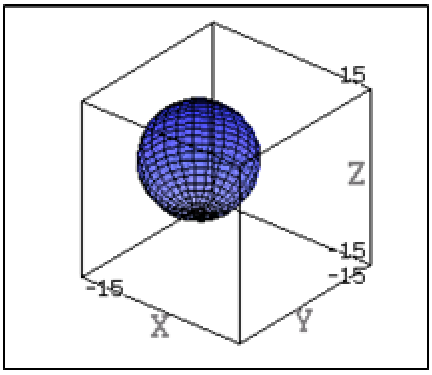
$$\frac{S_{GHIJKL}}{S_{ABCDEF}} = \left(\frac{\overline{LG}}{\overline{AB}}\right)^2 = \frac{7}{9}.$$

Generalization: In a regular hexagon ABCDEF the regular hexagon GHIJKL has been inscribed such that $\overline{AG} = k\cdot\overline{AB}$. Calculate the ratio between the areas of the two hexagons.

Solution:

$$\frac{S_{GHIJKL}}{S_{ABCDEF}} = k^2 - k + 1.$$

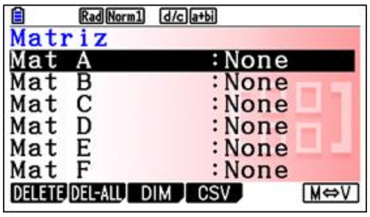
July 11: Find the equation of the sphere passing through the points $A(1,-2,-1), B(-5,10,-1), C(4,1,11), D(-8,-2,2)$

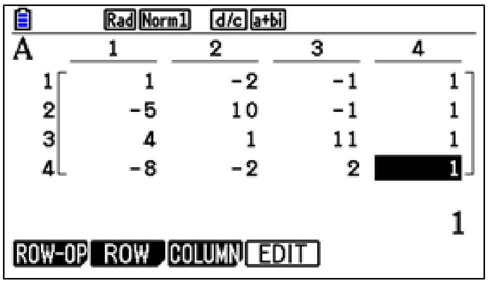


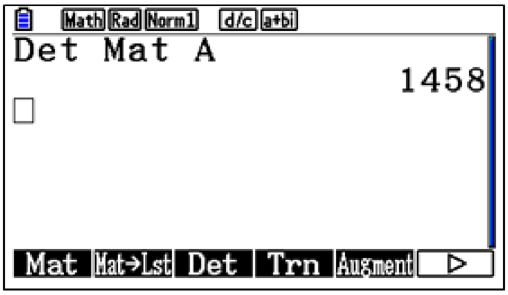
Solution 1: Let us see that the four points are not coplanar. We open the *Menu Exec-Mate*. We see that the determinant of the matrix

$$M = \begin{pmatrix} 1 & -2 & -1 & 1 \\ -5 & 10 & -1 & 1 \\ 4 & 1 & 11 & 1 \\ -8 & -2 & 2 & 1 \end{pmatrix}$$

is different from zero.







Let's calculate the midpoints of the segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$

The midpoint of the segment $\overline{AB}$ is $E(-2,4,-1)$

The midpoint of the segment $\overline{AC}$ is $F\left(\frac{5}{2},\frac{-1}{2},5\right)$

The midpoint of the segment $\overline{AD}$ is $G\left(\frac{-7}{2},-2,\frac{1}{2}\right)$

We calculate the components of the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$

$$\overrightarrow{AB} = (-6,-12,0)$$

$$\overrightarrow{AC} = (3,3,12)$$

$$\overrightarrow{AD} = (-9,0,3)$$

The median plane of the segment $\overline{AB}$ have equation:

$$\Pi_1 \equiv (x+2) + 2(y-4) = 0$$

$$\Pi_1 \equiv x + 2y = 6$$

The median plane of the segment $\overline{AC}$ have equation:

$$\Pi_2 \equiv \left(x-\frac{5}{2}\right) + \left(y+\frac{1}{2}\right) + 4(z-5) = 0$$

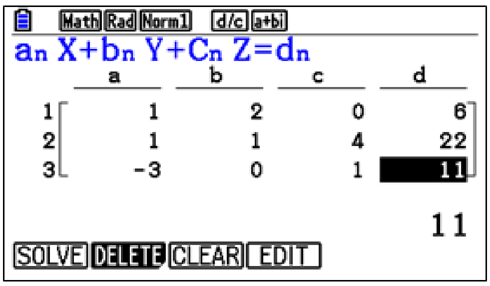
$$\Pi_2 \equiv x + y + 4z = 22$$

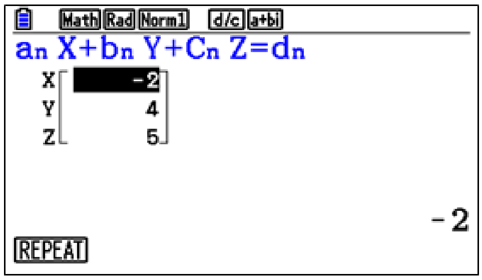
The median plane of the segment $\overline{AD}$ have equation:

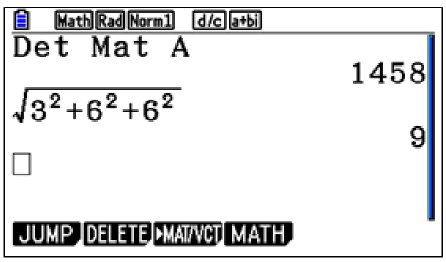
$$\Pi_3 \equiv -3\left(x+\frac{7}{2}\right) + \left(z-\frac{1}{2}\right) = 0$$

$$\Pi_3 \equiv -3x + z = 11$$

The center of the sphere is equal to the intersection point of the three planes. We open the *Menú Ecuación*:







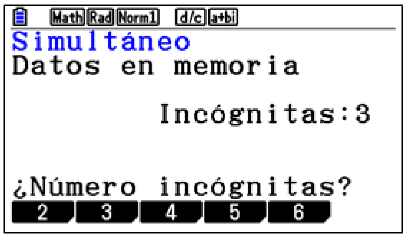
The center of the sphere is the point O(-2, 4, 5). The radius of the sphere is:

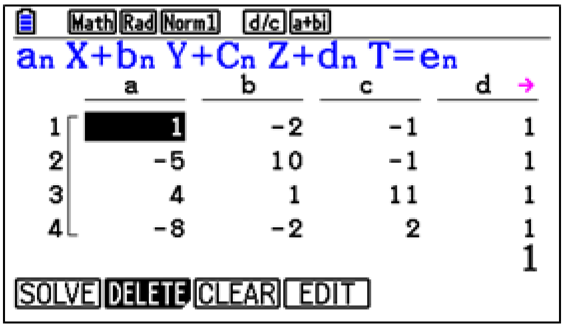
$$r = \overline{OA} = \sqrt{(1+2)^2 + (-2-4)^2 + (-1-5)^2}$$

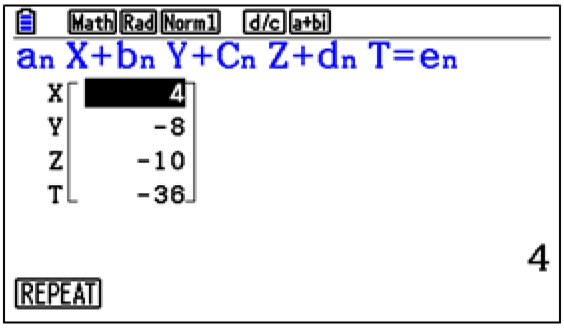
Solution 2: The general equation of the sphere is:

$$x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$$

We substitute the coordinates of the four points in the general equation and solve the system formed by the 4 equations with the unknowns A, B, C, D. We open the *Menú Ecuación*:







The solution is:

$$\begin{cases} A = 4 \\ B = -8 \\ C = -10 \\ D = -36 \end{cases}$$

The general equation of the sphere is:

$$x^2 + y^2 + z^2 + 4x - 8y - 10z + 36 = 0$$

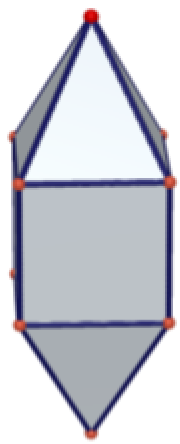
completing squares:

$$(x+2)^2 + (y-4)^2 + (z-5)^2 = 36 + 2^2 + 4^2 + 5^2; \qquad (x+2)^2 + (y-4)^2 + (z-5)^2 = 9^2$$

The center of the sphere is the point O(-2, 4, 5) and the radius $r = 9$

July 12: The figure is formed by a cube of edge a and two regular quadrangular pyramids of height a.

Determine the area and volume of the figure.



Solution: The volume is equal to the volume of the cube with edge a and of two square-based pyramids with edge a and height a.

$$V = a^3 + 2\left(\frac{1}{3}a^2 \cdot a\right) = \frac{5}{3}a^3$$

The apothem x of a face of the pyramids is the hypotenuse of a right triangle with legs 1/2 a and a. Applying the Pythagorean theorem:

$$x = \frac{\sqrt{5}}{2}a$$

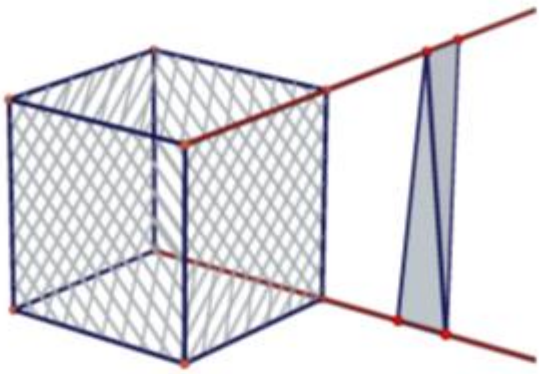
The area of the figure is equal to the sum of 4 squares with side a and 8 triangles with base a and height apothem $x = \frac{\sqrt{5}}{2}a$.

$$S = 4 \cdot a^2 + 8 \cdot \left(\frac{1}{2}a \cdot \frac{\sqrt{5}}{2}a\right) = \left(4 + 2\sqrt{5}\right)a^2$$

July 14-15: Two edges that are created from a cube extend.

In each extension a segment of a unit is taken.

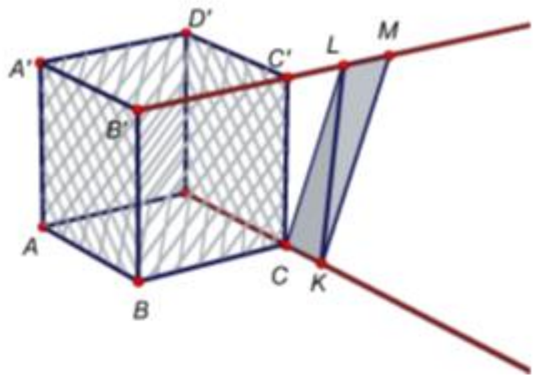
Where do the segments have to be located so that the volume of the tetrahedron formed by the 4 ends of the segment is maximum?.



Solution: Let ABCDA'B'C'D be the cube with edge $\overline{AB} = a$ with the following Cartesian coordinates:

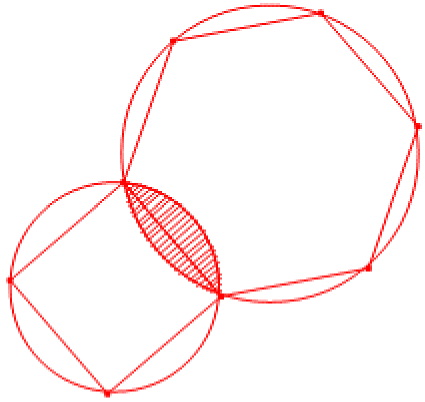
$$A(0,0,0),\ B(0,a,0),\ C(a,a,0),\ D(a,0,0),\ A'(0,0,a),$$
$$B'(0,a,a),\ C'(a,a,a),\ D'(a,0,a)$$

We can assume without losing generality that C is a vertex of the tetrahedron. Be $K(a, a+1, 0)$, $L(x, a, a)$, $M(x+1, a, a)$. We calculate the volume of the CKLM tetrahedron:



$$V_{CKLM} = \frac{1}{6}\left|\left[\overrightarrow{CK}\quad \overrightarrow{CL}\quad \overrightarrow{CM}\right]\right|.\quad V_{CKLM} = \frac{1}{6}\begin{vmatrix} 0 & 1 & 0 \\ x-a & 0 & a \\ x+1-a & 0 & a \end{vmatrix} = \frac{a}{6}.$$

The volume does not depend on the position of the two segments.

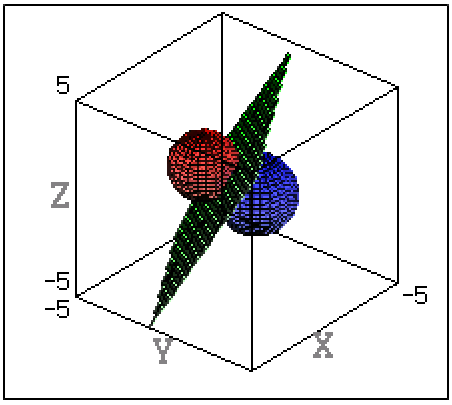


July 18-25: On one side of a regular hexagon of side c a square has been drawn (see figure).

Calculate the area of the intersection of the circles circumscribed to the two regular polygons.

Solution: The area is equal to the sum of the areas of two circular segments, one of radius c and angle 60° and one of radius $\frac{\sqrt{2}}{2}c$ and angle 90°.

$$S = \left(\frac{1}{6}\pi c^2 - \frac{\sqrt{3}}{4}c^2\right) + \left(\frac{1}{4}\pi\left(\frac{\sqrt{2}}{2}c\right)^2 - \frac{1}{4}c^2\right) = \frac{7\pi - 6\sqrt{3} - 6}{24}c^2.$$



July 19-20: Given the spheres

$$E_1 \equiv 2x^2 + 2y^2 + 2z^2 + 3x - 2y + z - 5 = 0,$$

$$E_2 \equiv x^2 + y^2 + z^2 - x + 3y - 2z + 1 = 0$$

Determine the relative position of the two spheres.

If they are intersecting, determine the plane where they intersect.

Determine the center and radius of the intersection circle of the two spheres.

Solution: completing squares:

$$E_1 \equiv \left(x + \frac{3}{4}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z + \frac{1}{4}\right)^2 = \frac{5}{2} + \frac{3}{16} + \frac{1}{4} + \frac{1}{16} = \frac{54}{16}$$

The center has coordinates, $O_1\left(-\frac{3}{4}, \frac{1}{2}, -\frac{1}{4}\right)$ and the radius is $R_1 = \frac{3}{4}\sqrt{6}$

$$E_2 \equiv \left(x - \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 + (z-1)^2 = -1 + \frac{1}{4} + \frac{9}{4} + 1 = \frac{10}{4}$$

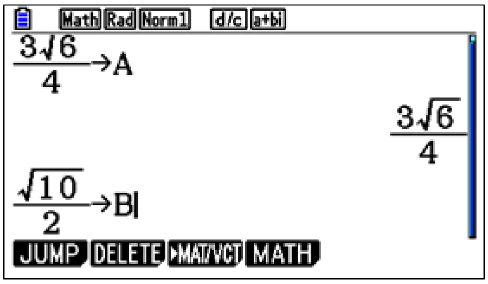
The center has coordinates, $O_2\left(\frac{1}{2}, -\frac{3}{2}, 1\right)$ and the radius is $R_1 = \frac{1}{2}\sqrt{10}$

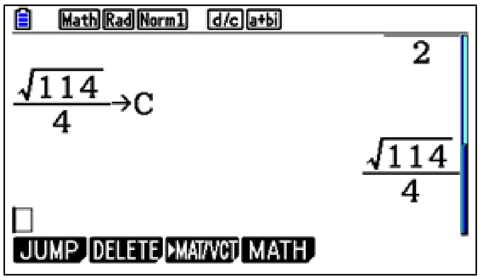
$$\overrightarrow{O_1O_2} = \left(\frac{5}{4}, 2, \frac{5}{4}\right)$$

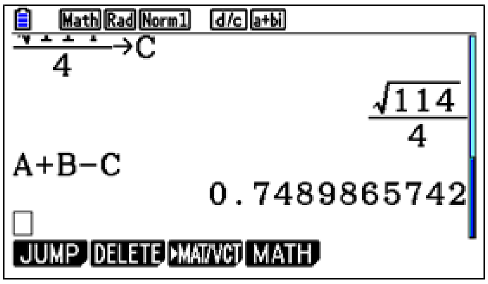
The distance between the centers is:

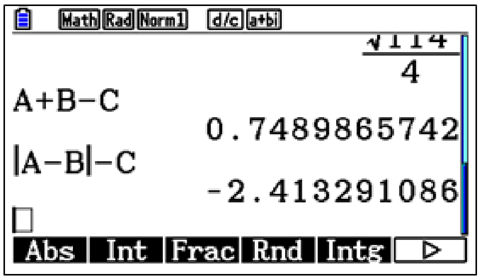
$$\overline{O_1O_2} = \frac{\sqrt{114}}{4}\overline{O_1O_2} = \frac{\sqrt{114}}{4} < R_1 + R_2 = \frac{3\sqrt{6} + 2\sqrt{10}}{4}$$

$$\overline{O_1O_2} = \frac{\sqrt{114}}{4} > |R_1 - R_2| = \frac{3\sqrt{6} - 2\sqrt{10}}{4}$$







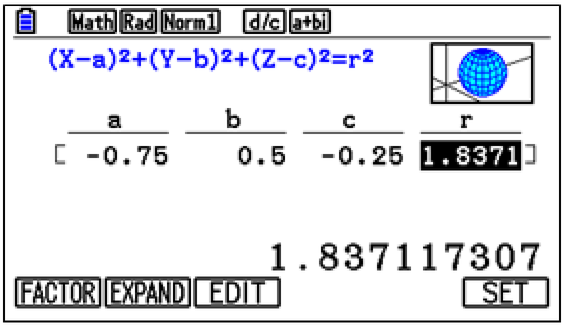


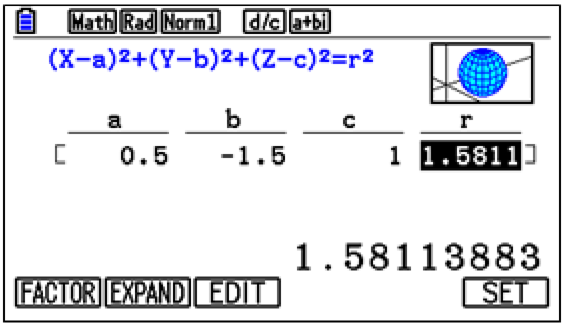
So the two spheres are secant.

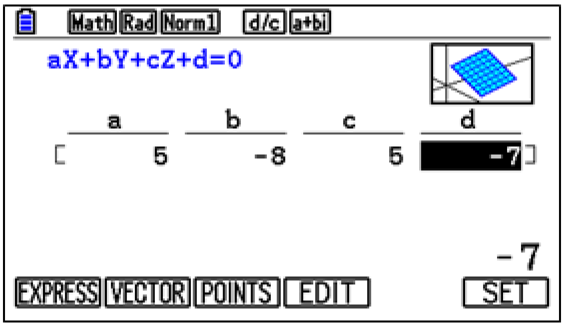
Let's calculate $E_1 - 2 \cdot E_2$ which gives us the intersection plane of the two spheres.

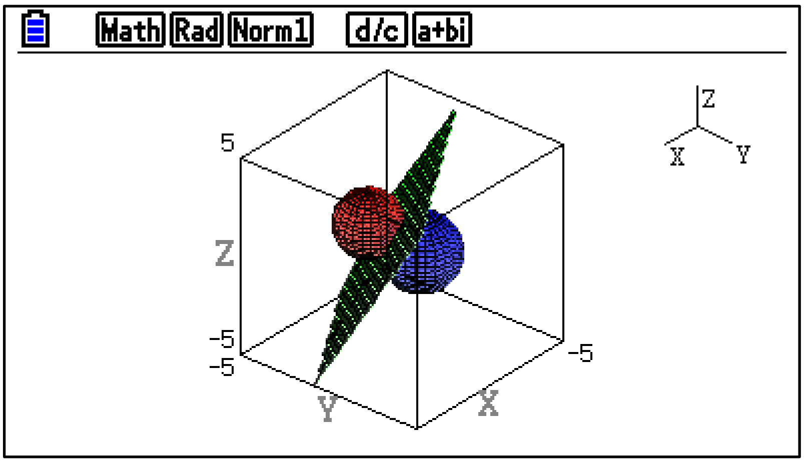
$$E_1 - 2 \cdot E_2 \equiv 5x - 8y + 5z - 7 = 0$$

We open the *Menú Gráfico 3D*. We define and represent the two spheres and the intersection plane.

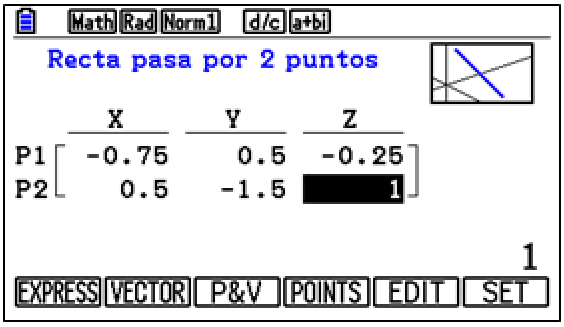


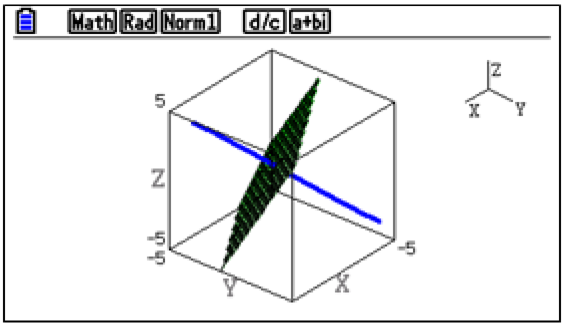




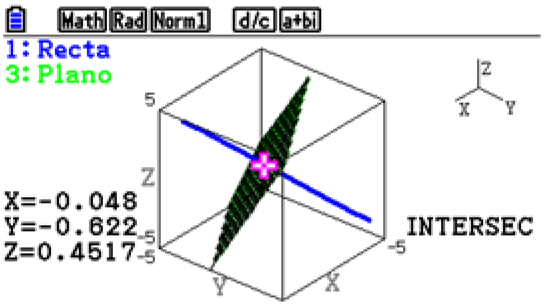


We define and represent the line that passes through the centers $O_1\left(-\frac{3}{4}, \frac{1}{2}, -\frac{1}{4}\right)$, $O_2\left(\frac{1}{2}, -\frac{3}{2}, 1\right)$





The center of the intersection circle is calculated by making the intersection of the line and the plane. With the *G-Solv* function, we determine the intersection:

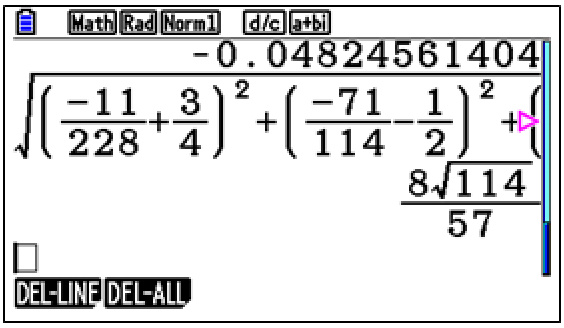


The center of the circle is:

$$O\left(\frac{-11}{228},\frac{-71}{114},\frac{103}{228}\right)$$

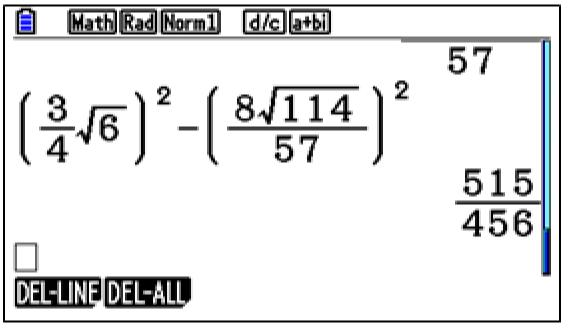
Let's calculate

$$\overline{O_1O}=\sqrt{\left(\frac{-11}{228}+\frac{3}{4}\right)^2+\left(\frac{-71}{114}+\frac{1}{2}\right)^2+\left(\frac{103}{228}+\frac{1}{4}\right)^2}$$
$$=\frac{8\sqrt{114}}{57}$$



Let R be the radius of the intersection circle of the two spheres. To calculate the radius R we will apply the Pythagorean theorem to the right triangle of legs, R, $\overline{O_1O}=\frac{8\sqrt{114}}{57}$ and hypotenuse $R_1=\frac{3}{4}\sqrt{6}$

$$\left(\frac{3}{4}\sqrt{6}\right)^2=R^2+\left(\frac{8\sqrt{114}}{57}\right)^2\ ;\ R^2=\frac{515}{456}$$



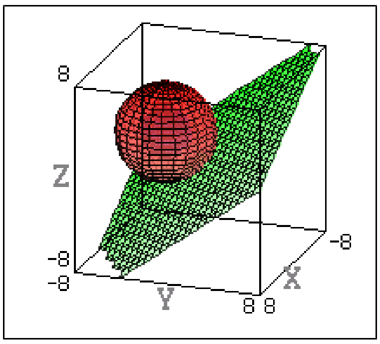
The radius of the circle is $R=\sqrt{\frac{515}{456}}$

<u>July 21-28:</u> Be the sphere

$$\mathbf{x^2+y^2+z^2+6y-4z+9=0}$$

Determine the equation of the sphere concentric with it that is tangent to the plane

$$\mathbf{2x-3y+2z+4=0}$$



Solution: Completing squares on the sphere $x^2 + y^2 + z^2 + 6y - 4z + 9 = 0$

$$x^2 + (y + 3)^2 + (z - 2)^2 = 2^2$$

The center of the sphere is the point O(0, -3, 2) and the radius $r = 2$

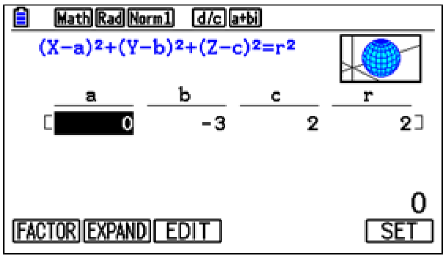
The radius of the concentric sphere tangent to the plane $\Pi \equiv 2x - 3y + 2z + 4 = 0$ is

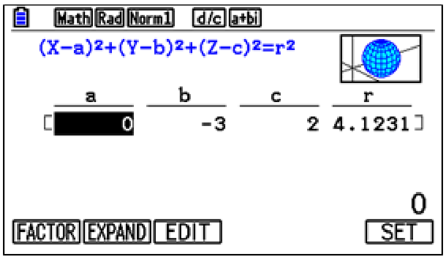
$$R = d(O, \Pi); \quad R = d(O, \Pi) = \left| \frac{2 \cdot 0 - 3 \cdot (-3) + 2 \cdot 2 + 4}{\sqrt{2^2 + (-3)^2 + 2^2}} \right| = \sqrt{17}$$

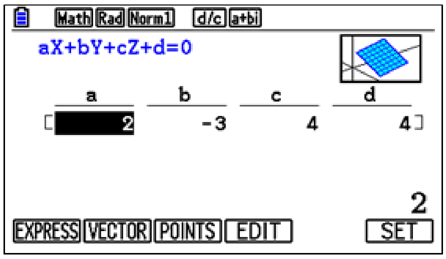
The equation of the sphere is:

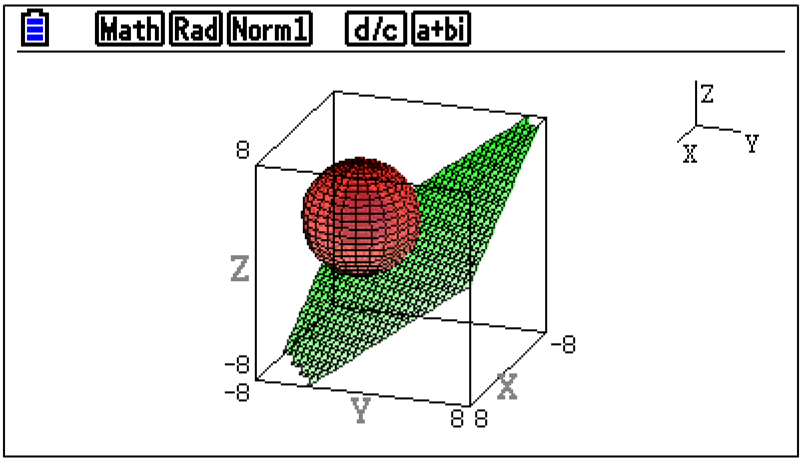
$$x^2 + (y + 3)^2 + (z - 2)^2 = 17$$

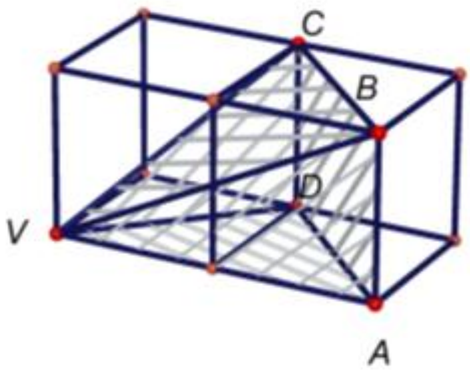
We open the *Menú Gráfico 3D*. We define and represent the two spheres and the plane











July 22: Let them be two equal cubes joined by a common face (see figure).

Determine the ratio between the volume of the pyramid ABCDV and the sum of the volumes of the two cubes.

Solution: Let $\overline{CD} = a$, be the edge of the two cubes. The sum of the volumes of the two cubes is:

$$V_{2c} = 2a^3$$

The rectangle ABCD is the base of the pyramid ABCDV.

$$\overline{VC} = a\sqrt{3}, \overline{CD} = a, \overline{VD} = a\sqrt{2}$$

Applying the inverse Pythagorean theorem, the triangle $\overset{\Delta}{VDC}$ is right $\angle VDC = 90°$

$$\overline{VD} = a\sqrt{2}, \qquad \overline{AD} = a\sqrt{2}, \qquad \overline{VA} = 2a$$

Applying the inverse Pythagorean theorem, the triangle $\overset{\Delta}{VDA}$ is right $\angle VDA = 90°$. So, $\overline{VD}$ is perpendicular to the base, and is the height of the pyramid. The volume of the pyramid is:

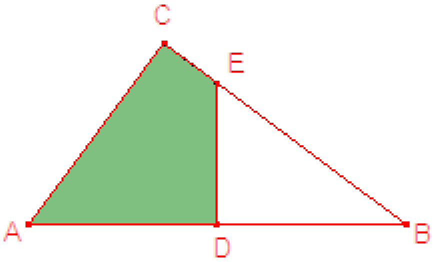
$$V_{ABCDV} = \frac{1}{3}\overline{DA}\cdot\overline{DC}\cdot\overline{VD} = \frac{1}{3}a\sqrt{2}\cdot a\cdot a\sqrt{2} = \frac{2}{3}a^3$$

The ratio between the volumes is:

$$\frac{V_{ABCDV}}{V_{2c}} = \frac{\frac{2}{3}a^3}{2a^3} = \frac{1}{3}$$

July 23: In the figure, $\overset{\Delta}{ABC}$ it is a right triangle $\angle C = 90^o$, $\overline{AD} = \overline{BD}$ and $\overline{DE} \perp \overline{AB}$.

If $\overline{AC} = 12$ and $AB = 20$, calculate the area of quadrilateral ADEC.



Solution: We have:

$$\overline{AD} = \overline{BD} = 10\,.$$

Applying the Pythagorean theorem to the right triangle $\overset{\Delta}{ABC}$:

$$\overline{BC} = 16\,.$$

Right triangles $\overset{\Delta}{ABC}$, $\overset{\Delta}{EBD}$ are similar. Applying Thales' theorem:

$$\frac{\overline{DE}}{10} = \frac{12}{16}. \text{ Then, } \overline{DE} = \frac{15}{2}. \quad \frac{\overline{BE}}{10} = \frac{20}{16}. \text{ Then, } \overline{BE} = \frac{25}{2}. \quad \overline{CE} = \overline{BC} - \overline{BE} = 16 - \frac{25}{2} = \frac{7}{2}.$$

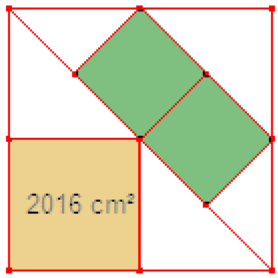
The area of quadrilateral ADEC is equal to the sum of the areas of the right triangles $\overset{\Delta}{ADE}$ and $\overset{\Delta}{ACE}$:

$$S_{ADEC} = \frac{1}{2}\cdot 10\cdot\frac{15}{2} + \frac{1}{2}\cdot 12\cdot\frac{7}{2} = \frac{117}{2} = 58.5\,.$$

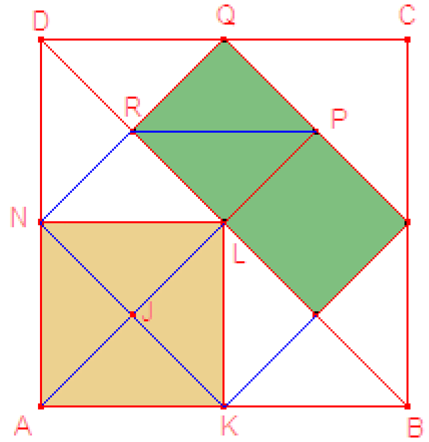
July 26-27: A square has been divided in two by the diagonal.

In the lower part a square of area $2016\ cm^2$ has been inscribed and in the upper part two identical small squares have been inscribed.

What is the area of each of the two small squares?.



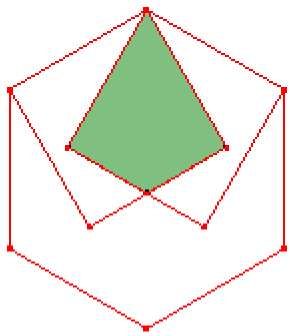
Solution:



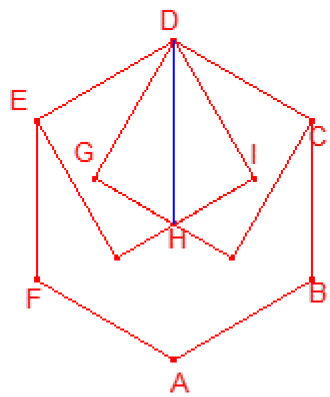
$$S_{LPQR} = \frac{1}{2} S_{AKLN} = 1008 \text{ cm}^2 .$$

July 29-30: On two consecutive sides of a regular hexagon, two squares have been drawn towards the interior of the hexagon.

Determine the ratio between the area of the intersection of the two squares and the area of the regular hexagon.



Solution: Let ABCDEF be the regular hexagon of side 1.



$$\overline{DG} = \overline{AB} = 1$$

Let DGHI be the intersection of the two squares.

$$\angle EDG = \angle EDC - \angle GDC = 120^o - 90^o = 30^o . \quad \angle GDI = \angle EDC - 2\angle EDG = 60^o .$$

$$\angle GDH = \frac{1}{2} \angle GDI = 30^o . \quad \overline{GH} = \frac{\sqrt{3}}{3} \overline{DG} = \frac{\sqrt{3}}{3} .$$

The area of the intersection of the two squares is:

$$S_{DGHI} = \overline{GH} \cdot \overline{DG} = \frac{\sqrt{3}}{3} .$$

The area of the regular hexagon is:

$$S_{ABCDEF} = 6\left(\frac{\sqrt{3}}{4} 1^2\right) = \frac{3\sqrt{3}}{2} .$$

The ratio between the two areas is:

$$\frac{S_{DGHI}}{S_{ABCDEF}} = \frac{\frac{\sqrt{3}}{3}}{\frac{3\sqrt{3}}{2}} = \frac{2}{9} .$$